

# Rigid Polyboxes and Keller's Conjecture

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## Abstract

A *cube tiling* of  $\mathbb{R}^d$  is a family of pairwise disjoint cubes  $[0, 1]^d + T = \{[0, 1]^d + t : t \in T\}$  such that  $\bigcup_{t \in T} ([0, 1]^d + t) = \mathbb{R}^d$ . Two cubes  $[0, 1]^d + t$ ,  $[0, 1]^d + s$  are called a *twin pair* if  $|t_j - s_j| = 1$  for some  $j \in [d] = \{1, \dots, d\}$  and  $t_i = s_i$  for every  $i \in [d] \setminus \{j\}$ . In 1930, Keller conjectured that in every cube tiling of  $\mathbb{R}^d$  there is a twin pair. Keller's conjecture is true for dimensions  $d \leq 6$  and false for all dimensions  $d \geq 8$ . For  $d = 7$  the conjecture is still open. Let  $x \in \mathbb{R}^d$ ,  $i \in [d]$ , and let  $L(T, x, i)$  be the set of all  $i$ th coordinates  $t_i$  of vectors  $t \in T$  such that  $([0, 1]^d + t) \cap ([0, 1]^d + x) \neq \emptyset$  and  $t_i \leq x_i$ . It is known that if  $|L(T, x, i)| \leq 2$  for some  $x \in \mathbb{R}^7$  and every  $i \in [7]$ , then Keller's conjecture is true for  $d = 7$ . In the present paper we show that it is also true for  $d = 7$  if  $|L(T, x, i)| \geq 6$  for some  $x \in \mathbb{R}^7$  and  $i \in [7]$ . Thus, if there is a counterexample to Keller's conjecture in dimension seven, then  $|L(T, x, i)| \in \{3, 4, 5\}$  for some  $x \in \mathbb{R}^7$  and  $i \in [7]$ .

*Key words:* box, cube tiling, rigidity, Keller's conjecture.

## 1 Introduction

A *cube tiling* of  $\mathbb{R}^d$  is a family of pairwise disjoint cubes  $[0, 1]^d + T = \{[0, 1]^d + t : t \in T\}$  such that  $\bigcup_{t \in T} ([0, 1]^d + t) = \mathbb{R}^d$ . Two cubes  $[0, 1]^d + t$ ,  $[0, 1]^d + s$  are called a *twin pair* if  $|t_j - s_j| = 1$  for some  $j \in [d] = \{1, \dots, d\}$  and  $t_i = s_i$  for every  $i \in [d] \setminus \{j\}$ . In 1907, Minkowski [18] conjectured that in every *lattice* cube tiling of  $\mathbb{R}^d$ , i.e. when  $T$  is a lattice in  $\mathbb{R}^d$ , there is a twin pair, and in 1930, Keller [8] generalized this conjecture to any cube tiling of  $\mathbb{R}^d$ . Minkowski's conjecture was confirmed by Hajós [7] in 1941. In 1940, Perron [19] proved that Keller's conjecture is true for all dimensions  $d \leq 6$ . In 1986, Szabó [21] showed that if there is a counterexample to Keller's conjecture in dimension  $d$ , then there is a counterexample two-periodic cube tiling  $[0, 1]^n + T$  of  $\mathbb{R}^n$ , where  $T \subset (1/2)\mathbb{Z}^n$  and  $d \leq n$ . Moreover, Corrádi and Szabó [3] reduced Keller's conjecture to a problem in graph theory. They defined a *d-dimensional Keller graph* whose vertices are all strings from the set  $\{0, 1, 2, 3\}^d$ . Two vertices are adjacent if they differ in at least two positions, but in at one position the difference is two modulo four. Keller's conjecture in the graph approach says that the maximal clique in a *d-dimensional Keller graph* has less than  $2^d$  vertices. The results of Corrádi and Szabó inspired Lagarias and Shor [13] who, in 1992, constructed a cube tiling of  $\mathbb{R}^{10}$  which does not contain a twin pair and thereby refuted Keller's

cube tiling conjecture. Finally, in 2002, Mackey [17] gave a counterexample to Keller's conjecture in dimension eight, which also shows that this conjecture is false in dimension nine. For  $d = 7$  Keller's conjecture is still open.

Let  $[0, 1]^d + T$  be a cube tiling,  $x \in \mathbb{R}^d$  and  $i \in [d]$ , and let  $L(T, x, i)$  be the set of all  $i$ th coordinates  $t_i$  of vectors  $t \in T$  such that  $([0, 1]^d + t) \cap ([0, 1]^d + x) \neq \emptyset$  and  $t_i \leq x_i$ . For every cube tiling  $[0, 1]^d + T$ ,  $x \in \mathbb{R}^d$  and  $i \in [d]$  the set  $L(T, x, i)$  contains at most  $2^{d-1}$  elements.

In 2010, Debroni et al. [4] computed that the maximal clique in a 7-dimensional Keller graph has 124 vertices, which implies that Keller's conjecture is true for all cube tilings  $[0, 1]^7 + T$  of  $\mathbb{R}^7$  with  $T \subset (1/2)\mathbb{Z}^7$  or equivalently,  $T \subset a + \mathbb{Z}^7 \cup b + \mathbb{Z}^7$ , where fixed  $a, b \in [0, 1]^7$  are such that  $a_i \neq b_i$  for every  $i \in [7]$ . The condition  $|L(T, x, i)| \leq 2$  for every  $i \in [7]$  means simply that  $T_1 \subset a + \mathbb{Z}^7 \cup b + \mathbb{Z}^7$ , where  $T_1 \subset T$  consists of all  $t$  for which  $([0, 1]^7 + t) \cap ([0, 1]^7 + x) \neq \emptyset$ . Thus, it is easy to show that the result of Debroni et al. proves also that the conjecture is true for cube tilings of  $\mathbb{R}^7$  with  $|L(T, x, i)| \leq 2$  for some  $x \in \mathbb{R}^7$  and every  $i \in [7]$ . Indeed, if there is no twin pairs in the set  $\{[0, 1]^7 + t : t \in T_1\}$ , then extending this family to the two-periodic tiling  $[0, 1]^7 + T$  of  $\mathbb{R}^7$ , where  $T = T_1 + 2\mathbb{Z}^7$ , we obtain a cube tiling with  $T \subset a + \mathbb{Z}^7 \cup b + \mathbb{Z}^7$  without twin pairs, which contradicts the result of Debroni et al.

In this paper we show that Keller's conjecture is true for cube tilings of  $\mathbb{R}^7$  with  $|L(T, x, i)| \geq 6$  for some  $x \in \mathbb{R}^7$  and  $i \in [7]$ . Thus, if there is a counterexample to Keller's conjecture in dimension seven, then  $|L(T, x, i)| \in \{3, 4, 5\}$  for some  $x \in \mathbb{R}^7$  and  $i \in [7]$ .

Similarly like Perron in [19] (see also [15]), we based our methods on the knowledge of the local structure of cube tilings. The main difference between Perron's approach and ours is that we use the notion of a rigid system of boxes, which was introduced in [14], and deeper examined in [10]. Roughly speaking, Perron's methods are more combinatorial, while ours are strongly geometric.

Works on Minkowski's and Keller's conjectures revealed a number of interesting problems concerning the structure of polybox codes and cube tilings. Lagarias and Shor [14] formulated a new problem on the structure of cube tilings of  $\mathbb{R}^d$ : let  $K_d$  be the largest integer such that every cube tiling of  $\mathbb{R}^d$  contains two cubes that have a common face of dimension  $K_d$ . What is the upper bound on  $K_d$ ? Since Keller's conjecture is true for  $d \leq 6$ , we have  $K_d = d - 1$  for  $d \leq 6$ . Generally, in [14] it was shown that  $K_d \leq d - 1/3\sqrt{d}$  for every  $d$ . Moreover, in that paper the authors considered subsets of  $\mathbb{R}^d$  which can be represent as a union of unit cubes, which satisfy some extra condition, in only a unique way. This leads to the notion of a rigid system of boxes, which is, as we have just mentioned, intensively exploit in the present paper.

A new approach to Minkowski's conjecture can be found in Kolountzakis's paper [12] and Kisielewicz's and Przesławski's paper [11]. The Hajos's proof of Minkowski's conjecture stimulates the development work on the factorization of abelian groups. These issues are examined in Szabo's book [22]. A fine survey of tilings of  $\mathbb{R}^d$  by clusters of unit cubes and Minkowski's conjecture is Stein's and Szabo's book [20].

It is not widely known that Keller made two conjectures on twin pairs in a cube tiling of  $\mathbb{R}^d$ . The second conjecture, posed in [9], says that every cube tiling of  $\mathbb{R}^d$  contains a *column* of unit cubes, i.e. a family of the form  $\{[0, 1]^d + t + ne_i : n \in \mathbb{Z}\}$ , where  $e_i$  is the  $i$ -th element of the standard basis of  $\mathbb{R}^d$ . This conjecture

has been proved by Lysakowska and Przesławski in [15]. Furthermore, these authors, in [16], described the meta-structure of cube tilings of  $\mathbb{R}^3$  and non-extensible systems of unit cubes. Such systems were also examined, among other things, by Dutour Sikirić and Itoh in [5].

The present paper is organized in the following way. In Section 2 we give basic notions concerning the systems of boxes and abstract words. These issues were developed in [6, 10]. Since they are not widely known, we present them in detail. In Section 3 we give results on the structure of special kind systems of boxes and partitions of a box into boxes. Next, we prove the fundamental for our purposes result on systems of words (Theorem 3.1). This theorem implies immediately the announced above result on Keller's conjecture in dimension seven, which we show in the final Section 4. At the end of the paper we extend the definition of a  $d$ -dimensional Keller graph and give an interpretation of our result in the terms of that graph.

## 2 Basic notions

In this section we present the basic notions on dichotomous boxes and words (details can be found in [6, 10]). We start with systems of boxes.

A non-empty set  $K \subseteq X = X_1 \times \cdots \times X_d$  is called a *box* if  $K = K_1 \times \cdots \times K_d$  and  $K_i \subseteq X_i$  for each  $i \in [d]$ . By  $\text{Box}(X)$  we denote the set of all boxes in  $X$ . The set  $X$  will be called a *d-box*. The box  $K$  is said to be *proper* if  $K_i \neq X_i$  for each  $i \in [d]$ . Two boxes  $K$  and  $G$  in  $X$  are called *dichotomous* if there is  $i \in [d]$  such that  $K_i = X_i \setminus G_i$ . A *suit* is any collection of pairwise dichotomous boxes. A suit is *proper* if it consists of proper boxes. A non-empty set  $F \subseteq X$  is said to be a *polybox* if there is a suit  $\mathcal{F}$  for  $F$ , i.e. if  $\bigcup \mathcal{F} = F$ . The important property of proper suits is that, if  $\mathcal{F}$  and  $\mathcal{G}$  are proper suits for a polybox  $F$ , then  $|\mathcal{F}| = |\mathcal{G}|$ . Thus, we can define a *box number*  $|F|_0 =$  the number of boxes in any proper suit for  $F$ . A proper suit for a  $d$ -box  $X$  is called a *minimal partition* of  $X$ . Every minimal partition of a  $d$ -box has  $2^d$  boxes.

A family  $\mathcal{C} \subset \text{Box}(X)$  is called a *simple partition* of  $X$  if for every  $K, G \in \mathcal{C}$  and every  $i \in [d]$  we have  $K_i = G_i$  or, if  $G_i \neq X_i$ ,  $K_i = X_i \setminus G_i$  and  $\mathcal{C}$  is a suit for  $X$ .

Two boxes  $K, G \subset X$  are said to be a *twin pair* if  $K_i = X_i \setminus G_i$  for some  $i \in [d]$  and  $K_j = G_j$  for every  $j \in [d] \setminus \{i\}$ .

Every two cubes  $[0, 1)^d + t$  and  $[0, 1)^d + p$  in an arbitrary cube tiling  $[0, 1)^d + T$  of  $\mathbb{R}^d$  satisfy *Keller's condition*: there is  $i \in [d]$  such that  $t_i - p_i \in \mathbb{Z} \setminus \{0\}$ , where  $t_i$  and  $p_i$  are  $i$ th coordinates of the vectors  $t$  and  $p$  ([8]). For any cube  $[0, 1)^d + x$ , where  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ , the family  $\mathcal{F}_x = \{([0, 1)^d + x) \cap ([0, 1)^d + t) \neq \emptyset : t \in T\}$  is a partition of the cube  $[0, 1)^d + x$ , in which, because of Keller's condition, every two boxes  $K, G \in \mathcal{F}_x$  are dichotomous, i.e. there is  $i \in [d]$  such that  $K_i$  and  $G_i$  are disjoint and  $K_i \cup G_i = [0, 1) + x_i$ . Moreover, since cubes in cube tilings are half-open, every box  $K \in \mathcal{F}_x$  is proper, and consequently the family  $\mathcal{F}_x$  is a minimal partition. The structure of the partition  $\mathcal{F}_x$  reflects the local structure of the cube tiling  $[0, 1)^d + T$ . This is easy to show that a cube tiling  $[0, 1)^d + T$  contains a twin pair if and only if the partition  $\mathcal{F}_x$  contains a twin pair for some  $x \in \mathbb{R}^d$ .

In order to sketch our approach to the problem of the existence of twin pairs in a cube tiling of  $\mathbb{R}^d$ , we first describe the structure of minimal partitions. A

graph-theoretic description of this structure can be found in [2].

Let  $X$  be a  $d$ -box. A set  $F \subseteq X$  is called an  $i$ -cylinder if for every  $x_j \in X_j$ ,  $j \in [d] \setminus \{i\}$ , one has

$$l_i \cap F = l_i \quad \text{or} \quad l_i \cap F = \emptyset,$$

where  $l_i = \{x_1\} \times \cdots \times \{x_{i-1}\} \times X_i \times \{x_{i+1}\} \times \cdots \times \{x_d\}$ .

Let  $\mathcal{F}$  be a minimal partition and let  $A \subset X_i$  be a set such that there is a box  $K \in \mathcal{F}$  with  $K_i \in \{A, X_i \setminus A\}$ . Let

$$\mathcal{W}^{i,A} = \{K \in \mathcal{F} : K_i = A\} \quad \text{and} \quad \mathcal{W}^{i,A'} = \{K \in \mathcal{F} : K_i = X_i \setminus A\}.$$

Since boxes in  $\mathcal{F}$  are pairwise dichotomous, the set  $\bigcup(\mathcal{W}^{i,A} \cup \mathcal{W}^{i,A'})$  is an  $i$ -cylinder, and the set of boxes  $\mathcal{W}^{i,A} \cup \mathcal{W}^{i,A'}$  is a suit for it. As  $|\mathcal{F}| = 2^d$ , it follows that the boxes in  $\mathcal{F}$  can form at most  $2^{d-1}$  pairwise disjoint  $i$ -cylinders. More precisely, for every  $i \in [d]$  there are sets  $A^1, \dots, A^k \subset X_i$  such that  $A^n \notin \{A^m, X_i \setminus A^m\}$  for every  $n, m \in [k]$ ,  $n \neq m$ , and

$$\mathcal{F} = \mathcal{W}^{i,A^1} \cup \mathcal{W}^{i,(A^1)'} \cup \dots \cup \mathcal{W}^{i,A^k} \cup \mathcal{W}^{i,(A^k)'}$$

The boxes in  $\mathcal{F}$  are proper, and hence  $|\mathcal{W}^{i,A^n} \cup \mathcal{W}^{i,(A^n)'}| \geq 2$ . Thus  $k \leq 2^{d-1}$ . Observe that, that is why  $|L(T, x, i)| \leq 2^{d-1}$  for every cube tiling  $[0, 1)^d + T$ ,  $x \in \mathbb{R}^d$  and  $i \in [d]$ , as  $|L(T, x, i)|$  is the number of all  $i$ -cylinders in the partition  $\mathcal{F}_x$ .

If  $K$  is a box in  $X$  and  $\mathcal{G}$  is a family of boxes, then let

$$(K)_i = K_1 \times \cdots \times K_{i-1} \times K_{i+1} \times \cdots \times K_d \quad \text{and} \quad (\mathcal{G})_i = \{(K)_i : K \in \mathcal{G}\}.$$

Since  $\bigcup(\mathcal{W}^{i,A} \cup \mathcal{W}^{i,A'})$  is an  $i$ -cylinder, the sets of boxes  $(\mathcal{W}^{i,A})_i$  and  $(\mathcal{W}^{i,A'})_i$  are two suits for the polybox  $\bigcup(\mathcal{W}^{i,A})_i = \bigcup(\mathcal{W}^{i,A'})_i$ , which is a polybox in the  $(d-1)$ -box  $(X)_i$  (Figure 1). Note that, as  $(\mathcal{W}^{i,A})_i$  and  $(\mathcal{W}^{i,A'})_i$  are proper suits for the polybox  $\bigcup(\mathcal{W}^{i,A})_i$ , we have  $|(\mathcal{W}^{i,A})_i| = |(\mathcal{W}^{i,A'})_i|$

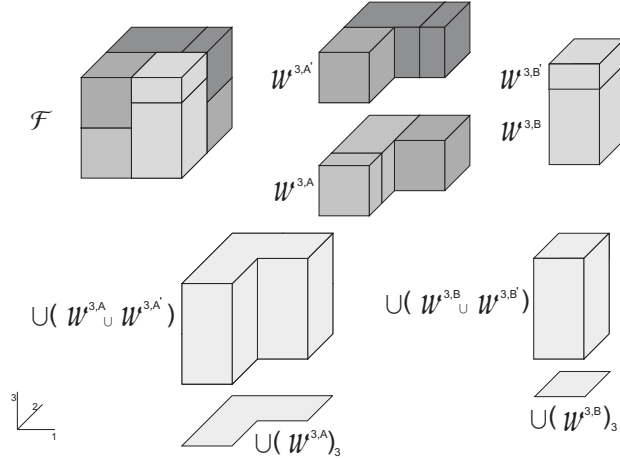


Fig. 1. The minimal partition  $\mathcal{F} = \mathcal{W}^{3,A} \cup \mathcal{W}^{3,A'} \cup \mathcal{W}^{3,B} \cup \mathcal{W}^{3,B'}$  of the 3-box  $X = [0, 1]^3$  ( $A = [0, 1/2)$ ,  $B = [0, 3/4)$ ), two 3-cylinders and its suits.

Now, if  $K, G \in \mathcal{F}$  are a twin pair, then there is a suit  $\mathcal{W}^{i,A} \cup \mathcal{W}^{i,A'} \subset \mathcal{F}$  for some  $i$ -cylinder such that  $K, G \in \mathcal{W}^{i,A} \cup \mathcal{W}^{i,A'}$ . Thus,  $K, G \in \mathcal{W}^{i,A}$  or  $K, G \in \mathcal{W}^{i,A'}$  or  $K \in \mathcal{W}^{i,A}$  and  $G \in \mathcal{W}^{i,A'}$ . In the third case  $(K)_i = (G)_i \in (\mathcal{W}^{i,A})_i \cap (\mathcal{W}^{i,A'})_i$ . So, if  $(\mathcal{W}^{i,A})_i \cap (\mathcal{W}^{i,A'})_i \neq \emptyset$ , then there is a twin pair in  $\mathcal{F}$  (see e.g. [2]). Now we can ask on the maximal positive integer  $n$  such that if  $\mathcal{W}^{i,A}$  and  $\mathcal{W}^{i,A'}$  do not contain a twin pair and  $|\mathcal{W}^{i,A}| \leq n$ , then  $(\mathcal{W}^{i,A})_i = (\mathcal{W}^{i,A'})_i$ . We will show that  $n = 11$ , from which the announced in Section 1 result on Keller's conjecture in dimension seven will easily follow.

The results in the present paper are formulated and proved in full generality. Suits have the form of systems of abstract words. We collect below basic notions concerning words (details can be found in [10]).

A set  $S$  of any objects will be called an *alphabet*, and the elements of  $S$  will be called *letters*. A permutation  $s \mapsto s'$  of the alphabet  $S$  such that  $s'' = (s')' = s$  and  $s' \neq s$  is said to be a *complementation*. We add an extra element  $*$  to the set  $S$  and the set  $S \cup \{*\}$  is denoted by  $*S$ . We set  $*' = *$ . Each sequence of letters  $s_1 \cdots s_d$  from the set  $*S$  is called a *word*. The set of all words of length  $d$  is denoted by  $(*S)^d$ , and by  $S^d$  we denote the set of all words  $s_1 \cdots s_d$  such that  $s_i \neq *$  for every  $i \in [d]$ . Two words  $u = u_1 \cdots u_d$  and  $v = v_1 \cdots v_d$  are *dichotomous* if there is  $j \in [d]$  such that  $u_j \neq *$  and  $u'_j = v_j$ . If  $V \subset (*S)^d$  consists of pairwise dichotomous words, then we call it a *polybox code* (or *polybox genome*). Two words  $u, v \in (*S)^d$  are a *twin pair* if there is  $j \in [d]$  such that  $u'_j = v_j$ , where  $u_j \neq *$  and  $u_i = v_i$  for every  $i \in [d] \setminus \{j\}$ .

If  $A \subset [d]$  and  $[d] \setminus A = \{i_1 < \cdots < i_n\}$ , then  $(u)_A = u_{i_1} \cdots u_{i_n}$  and  $(V)_A = \{(v)_A : v \in V\}$  for  $V \subset (*S)^d$ . If  $A = \{i\}$ , then we write  $(u)_i$  and  $(V)_i$  instead of  $(u)_{\{i\}}$  and  $(V)_{\{i\}}$ , respectively. If  $V \subset (*S)^d$ ,  $s \in *S$  and  $i \in [d]$ , then let  $V^{i,s} = \{v \in V : v_i = s\}$ .

Suppose now that for each  $i \in [d]$  a mapping  $f_i: *S \rightarrow \text{Box}(X_i)$  is such that  $f_i(s') = X_i \setminus f_i(s)$  for  $s \neq *$  and  $f_i(*) = X_i$ . We define the mapping  $f: (*S)^d \rightarrow \text{Box}(X)$  by

$$f(s_1 \cdots s_d) = f_1(s_1) \times \cdots \times f_d(s_d).$$

About such defined  $f$  we will say that it *preserves dichotomies*. If  $V \subseteq (*S)^d$ , then  $f(V) = \{f(v) : v \in V\}$  is said to be a *realization* of the set of words  $V$ . Clearly, if  $V$  is a polybox code, then  $f(V)$  is a suit for  $\bigcup f(V)$ . The realization is said to be *exact* if for each pair of words  $v, w \in V$ , if  $v_i \notin \{w_i, w'_i\}$ , then  $f_i(v_i) \notin \{f_i(w_i), X_i \setminus f_i(w_i)\}$ .

A polybox code  $V \subset (*S)^d$  is called a *partition code* if any realization  $f(V)$  of  $V$  is a suit for a  $d$ -box  $X$ . Observe that, if  $V \subset S^d$  is a partition code, then  $f(V)$  is a minimal partition. A partition code  $C \subset (*S)^d$  is said to be a *simple* if for every  $v, w \in C$  and every  $i \in [d]$  we have  $v_i = w_i$  or  $v_i = w'_i$ .

We will exploit some abstract but very useful realization of polybox codes. This sort of realization was invented in [1], where it was the crucial tool in proving the main theorem of that paper.

Let  $S$  be an alphabet with a complementation, and let

$$ES = \{B \subset S : |\{s, s'\} \cap B| = 1, \text{ whenever } s \in S\},$$

$$Es = \{B \in ES : s \in B\} \text{ and } E* = ES.$$

Let  $V \subset (*S)^d$  be a polybox code, and let  $v \in V$ . The *equicomplementary* realization of the word  $v$  is the box

$$\check{v} = Ev_1 \times \cdots \times Ev_d$$

in the  $d$ -box  $(ES)^d = ES \times \cdots \times ES$ . The equicomplementary realization of the code  $V$  is the family

$$E(V) = \{\check{v} : v \in V\}.$$

If  $S$  is finite,  $s_1, \dots, s_n \in S$  and  $s_i \notin \{s_j, s'_j\}$  for every  $i \neq j$ , then

$$|Es_1 \cap \cdots \cap Es_n| = (1/2^n)|ES|. \quad (2.1)$$

The value of the realization  $E(V)$ , where  $V \subset S^d$ , lies in the above equality. In particular, boxes in  $E(V)$  are of the same size; for  $w \in E(V)$  we have  $|\check{w}| = (1/2^d)|ES|^d$ . Thus, two boxes  $\check{v}, \check{w} \subset (ES)^d$  are dichotomous if and only if  $\check{v} \cap \check{w} = \emptyset$ .

Moreover, from (2.1) we obtain the following important lemma.

**LEMMA 2.1** *Let  $w, u, v \in S^d$ , and let  $\mathcal{D}$  be a simple partition of the  $d$ -box  $\check{w}$ . If boxes  $\check{w} \cap \check{u}$  and  $\check{w} \cap \check{v}$  belong to  $\mathcal{D}$ , then there is a simple partition code  $C \subset S^d$  such that  $u, v \in C$ . In particular, if  $\check{w} \cap \check{u}$  and  $\check{w} \cap \check{v}$  form a twin pair, then  $u$  and  $v$  are a twin pair.*

*Proof.* Assume on the contrary that  $u$  and  $v$  do not belong to any simple partition code  $C \subset S^d$ . Then there is  $j \in [d]$  such that  $u_j \notin \{v_j, v'_j\}$ . Since  $\check{w} \cap \check{u}, \check{w} \cap \check{v} \in \mathcal{D}$ , for every  $i \in [d]$  we have  $Ew_i \cap Eu_i = Ew_i \cap Ev_i$  or  $Ew_i \cap Eu_i = Ew_i \setminus Ew_i \cap Ev_i$ . If  $Ew_j \cap Eu_j = Ew_j \cap Ev_j$ , then  $Ew_j \cap Eu_j \cap Ev_j = Ew_j \cap Ev_j$ , which is impossible, as, by (2.1),  $|Ew_j \cap Eu_j \cap Ev_j| = (1/8)|ES|$ , while  $|Ew_j \cap Ev_j| = (1/4)|ES|$ . In the second case the non-empty sets  $Ew_i \cap Eu_i$  and  $Ew_i \cap Ev_i$  are disjoint, which is a contradiction, since, again by (2.1),  $|Ew_j \cap Eu_j \cap Ev_j| = (1/8)|ES|$ .

If  $\check{w} \cap \check{u}$  and  $\check{w} \cap \check{v}$  form a twin pair, then,  $Ew_j \cap Eu_j = Ew_j \setminus Ew_j \cap Ev_j$  for exactly one  $j \in [d]$  and  $Ew_i \cap Eu_i = Ew_i \cap Ev_i$  for every  $i \in [d] \setminus \{j\}$ . Therefore, by the first part of the lemma,  $u, v \in C$  for some simple partition code  $C \subset S^d$ . Thus,  $u_j = v'_j$  for  $j \in [d]$  and  $u_i = v_i$  for every  $i \in [d] \setminus \{j\}$ .  $\square$

Let  $V \subset (*S)^d$  be a partition code. From [Lemma 8.1, [10]] it follows that there is a simple partition code  $C \subset (*S)^d$  and there are two words  $v, w \in V \cap C$  such that

$$|\{i \in [d] : v_i = w'_i, v_i \neq *\}| \equiv 1 \pmod{2}. \quad (2.2)$$

Let  $V, W \subset (*S)^d$  be polybox codes, and let  $v \in (*S)^d$ . We say that  $v$  is *covered* by  $W$ , and write  $v \sqsubseteq W$ , if  $f(v) \subseteq \bigcup f(W)$  for every mapping  $f$  that preserves dichotomies. If  $v \sqsubseteq W$  for every  $v \in V$ , then we write  $V \sqsubseteq W$ .

Let  $g: S^d \times S^d \rightarrow \mathbb{Z}$  be defined by the formula

$$g(v, w) = \prod_{i=1}^d (2[v_i = w_i] + [w_i \notin \{v_i, v'_i\}]), \quad (2.3)$$

where  $[p] = 1$  if the sentence  $p$  is true and  $[p] = 0$  if it is false.

Let  $w \in S^d$ , and let  $V \subset S^d$  be a polybox code. Then

$$\check{w} \subseteq \bigcup E(V) \Leftrightarrow w \sqsubseteq V \Leftrightarrow \sum_{v \in V} g(v, w) = 2^d. \quad (2.4)$$

Let  $s_* = * \cdots * \in (*S)^d$  and let  $\bar{g}(\cdot, s_*): (*S)^d \rightarrow \mathbb{Z}$  be defined as follows:

$$\bar{g}(v, s_*) = \prod_{i=1}^d (2[v_i = *] + [v_i \neq *]).$$

LEMMA 2.2 *Let  $V \subset (*S)^d$ . The set  $V$  is a partition code if and only if  $\sum_{v \in V} \bar{g}(v, s_*) = 2^d$ .*

*Proof.* By (2.1), the definition of  $\bar{g}$  and the equality  $E* = ES$ , we have  $\bar{g}(v, s_*)|ES|^d/2^d = |\check{v}|$ . Since  $V$  is a partition code,  $\sum_{v \in V} |\check{v}| = |ES|^d$ , and then  $\sum_{v \in V} \bar{g}(v, s_*) = 2^d$ . If  $\sum_{v \in V} \bar{g}(v, s_*) = 2^d$ , then  $\sum_{v \in V} |\check{v}| = |ES|^d$ , which means that  $V$  is a partition code.  $\square$

COROLLARY 2.3 *Let  $V \subset S^d$  be a polybox code and let  $u \in S^d$ . For every  $v \in V$  let  $\bar{v} \in (*S)^d$  be defined in the following way: if  $v_i \neq u_i$ , then  $\bar{v}_i = v_i$ , and if  $v_i = u_i$ , then  $\bar{v}_i = *$ . Let  $\check{u} \cap \check{v} \neq \emptyset$  for every  $v \in V$ . If  $u \sqsubseteq V$ , then  $\bar{V} = \{\bar{v} : v \in V\}$  is a partition code.*

*Proof.* By (2.1),  $|\check{v} \cap \check{u}| = (1/2^k)(1/2^d)|ES|^d$ , where  $k = |\{i : v_i \neq u_i\}|$ , and by the definition of the function  $\bar{g}$ ,  $1/2^k = (1/2^d)\bar{g}(\bar{v}, s_*)$ . The set  $\{\check{u} \cap \check{v} : v \in V\}$  is a suit for the  $d$ -box  $\check{u}$ . Thus,  $\sum_{v \in V} |\check{v} \cap \check{u}| = |\check{u}|$ . Therefore,  $\sum_{\bar{v} \in \bar{V}} (1/2^d)\bar{g}(\bar{v}, s_*)(1/2^d)|ES|^d = (1/2^d)|ES|^d$ , which gives  $\sum_{\bar{v} \in \bar{V}} \bar{g}(\bar{v}, s_*) = 2^d$ . By Lemma 2.2,  $\bar{V}$  is a partition code.  $\square$

### 3 Equivalent polybox codes without twin pairs.

Polybox codes  $V, W \subset (*S)^d$  are said to be *equivalent* if  $V \sqsubseteq W$  and  $W \sqsubseteq V$ . Thus, if  $V$  and  $W$  are equivalent, then  $\bigcup f(V) = \bigcup f(W)$  for every function  $f$  preserving dichotomies. In particular,  $\bigcup E(V) = \bigcup E(W)$ . A polybox code  $V \subset S^d$  is called *rigid* if there is no code  $W \subset S^d$  which is equivalent to  $V$  and  $V \neq W$ .

Our result on Keller's conjecture in dimension seven is based on the following theorem.

THEOREM 3.1 *If  $V, W \subset S^d$  are equivalent polybox codes which do not contain twin pairs and  $W \cap V = \emptyset$ , then  $|V| \geq 12$ .*

To prove this theorem we need several auxiliary results. First, we will describe one polybox code and two partition codes without twin pairs which contain a few words. Next, we will prove special cases in which Theorem 3.1 holds.

If  $v = v_1 \cdots v_d \in (*S)^d$ ,  $V \subset (*S)^d$  and  $\sigma$  is a permutation of the set  $[d]$ , then  $v_\sigma = v_{\sigma(1)} \cdots v_{\sigma(d)}$  and  $V_\sigma = \{v_\sigma : v \in V\}$ .

Let

$$V = \{l_1 * o_3 \cdots o_d, l'_1 l_2 o_3 \cdots o_d\}, \quad (3.1)$$

where  $d \geq 2$ ,  $l_1, l_2 \in S$  and  $o_3, \dots, o_d \in *S$ . Let  $X$  be a  $d$ -box and  $F \subset X$  be a polybox.  $F$  is said to be a  $L$ -polybox if there is a permutation  $\sigma$  of the set  $[d]$



such that the set  $V_\sigma$  is the polybox code for  $F$ , where  $V$  is of the form (3.1) (Figure 2 and 3).

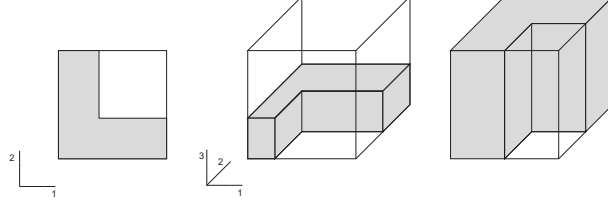


Fig. 2. A  $L$ -polybox in the two-dimensional case (on the left) and two  $L$ -polyboxes in the three-dimensional case: for  $o_3 \neq *$  (in the middle) and for  $o_3 = *$  (on the right).

**LEMMA 3.2** *Let  $X$  be a  $d$ -box, and let  $F \subset X$  be a polybox. Assume that  $\mathcal{V}$  and  $\mathcal{W}$  are two disjoint suits for  $F$  without twin pairs and  $|\mathcal{V}| = |\mathcal{W}| = 2$ . Then  $F$  is a  $L$ -polybox and the suits  $\mathcal{V}$  and  $\mathcal{W}$  are exact realizations of codes  $V_\sigma$  and  $W_\sigma$ , where  $V = \{l_1 * o_3 \cdots o_d, l'_1 l_2 o_3 \cdots o_d\} \subset (*S)^d$ ,  $W = \{l_1 l'_2 o_3 \cdots o_d, *l_2 o_3 \cdots o_d\} \subset (*S)^d$ ,  $l_1, l_2 \in S$ ,  $o_3, \dots, o_d \in *S$  and  $\sigma$  is a permutation of the set  $[d]$ . In particular, if  $X = \check{s}$ , where  $s \in S^d$ , then  $V = \{l_1 s_2 o_3 \cdots o_d, l'_1 l_2 o_3 \cdots o_d\} \subset (*S)^d$ ,  $W = \{l_1 l'_2 o_3 \cdots o_d, s_1 l_2 o_3 \cdots o_d\} \subset (*S)^d$ ,  $l_i \notin \{s_i, s'_i\}$  for  $i = 1, 2$  and  $o_i \neq s'_i$  for  $i \in \{3, \dots, d\}$ .*

*Proof.* Let  $W = \{v, u\} \subset (*S)^d$  and  $V = \{w, q\} \subset (*S)^d$  be polybox codes for  $F$ , which are exact realizations of  $\mathcal{W}$  and  $\mathcal{V}$ , respectively. By the assumptions, they are equivalent, disjoint and do not contain twin pairs. Assume, without loss of generality, that  $\check{w} \cap \check{v} \neq \emptyset$  and  $\check{q} \cap \check{v} \neq \emptyset$ . If the set  $\check{q} \setminus \check{v}$  is nonempty, then it is a box. If not, at least two additional words  $p, r \in W \setminus \{v\}$  are needed to cover the set  $\check{q} \setminus \check{v}$  by the boxes  $\check{p}, \check{r}$ , which contradicts the assumption on  $W$ .

Observe that, if  $A$  and  $B$  are boxes,  $A \cap B \neq \emptyset$  and  $A \setminus B \neq \emptyset$  is a box, then there is exactly one  $k \in [d]$  such that  $A_k \setminus B_k \neq \emptyset$  and  $A_n \subseteq B_n$  for every  $n \in [d] \setminus \{k\}$ .

Assume first that  $\check{q} \setminus \check{v} \neq \emptyset$  and  $\check{w} \setminus \check{v} \neq \emptyset$ . Since  $\check{v} \cup \check{u} = \check{q} \cup \check{w}$ , the set  $\check{q} \setminus \check{v} \cup \check{w} \setminus \check{v}$  is a box ( $= \check{u}$ ). Then the boxes  $\check{q} \setminus \check{v}$  and  $\check{w} \setminus \check{v}$  are a twin pair. On the other hand, the boxes  $\check{q} \cap \check{v}$  and  $\check{w} \cap \check{v}$  are also a twin pair, as  $\check{q} \cap \check{v} \cup \check{w} \cap \check{v} = \check{v}$ . Thus,  $w$  and  $q$  are a twin pair, a contradiction. So, let  $\check{q} \setminus \check{v} \neq \emptyset$  and  $\check{w} \setminus \check{v} = \emptyset$ . Then  $\check{w} \subset \check{v}$ . Since  $\check{q} \cap \check{v} \neq \emptyset$  and  $\check{w} \cap \check{v} \neq \emptyset$  there is  $i \in [d]$  such that  $(\check{q})_i \cap (\check{w})_i \neq \emptyset$ , and thus  $q_i = w'_i$ ,  $q_i \neq *$ . Then, by (2.1),  $v_i = *$ . Moreover,  $v_n = w_n$  for every  $n \in [d] \setminus \{i\}$ , as  $\check{v} \setminus \check{w}$  is a box and  $\check{w} \subset \check{v}$ .

We have  $\check{q} \cap \check{u} \neq \emptyset$ ,  $\check{q} \cap \check{v} \neq \emptyset$  and  $\check{v} \setminus \check{q} \neq \emptyset$ . For the same reasons as previously,  $\check{u} \subset \check{q}$ . Hence, there is  $j \in [d] \setminus \{i\}$  such that  $v_j = u'_j$ ,  $u_j \neq *$ ,  $q_j = *$  and  $q_n = u_n$  for every  $n \in [d] \setminus \{j\}$ .

Since  $\check{q} \setminus \check{u} = \check{v} \setminus \check{w}$ , we have  $q_n = w_n = u_n = v_n$  for every  $n \in [d] \setminus \{i, j\}$ ,  $q_n \in *S$ . Thus,  $(v)_A = *v_j$ ,  $(u)_A = q_i v'_j$  and  $(q)_A = q_i *$ ,  $(w)_A = q'_i v_j$ , where  $A = [d] \setminus \{i, j\}$  and  $q_i, v_j \in S$ .  $\square$



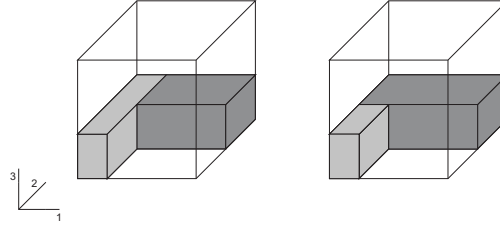


Fig. 3. Two realizations of polybox codes for a  $L$ -polybox:  $V = \{l_1 * o_3, l'_1 l_2 o_3\}$ ,  $o_3 \neq *$ , (on the left) and  $W = \{l_1 l'_2 o_3, * l_2 o_3\}$  (on the right).

LEMMA 3.3 *Let  $V \subset (*S)^d$  be a partition code without twin pairs.*

(a) *If  $|V| > 1$ , then  $d \geq 3$  and  $|V| \geq 5$ . The equality  $|V| = 5$  holds if and only if there are  $i_1, i_2, i_3 \in [d]$  such that*

$$(V)_A = \{l_1 l_2 l_3, l'_1 l'_2 l'_3, * l_2 l'_3, l'_1 * l_3, l_1 l'_2 *\}$$

*or*

$$(V)_A = \{l_1 l_2 l_3, l'_1 l'_2 l'_3, * l'_2 l_3, l_1 * l'_3, l'_1 l_2 *\},$$

*where  $A = [d] \setminus \{i_1, i_2, i_3\}$ ,  $l_1, l_2, l_3 \in S$  and  $v = * \cdots *$  for every  $v \in V_{\{i_1, i_2, i_3\}}$ .*

(b) *If  $|V| = 6$ , then  $d \geq 4$  and there are  $i \in [d]$  and  $l \in S$  such that*

$$V = V^{i,l} \cup V^{i,l'},$$

*where  $|V^{i,l}| = 1$  and  $|V^{i,l'}| = 5$ .*

*Proof of (a)* By (2.2) there is a simple partition code  $C \subset (*S)^d$  and there are words  $v, u \in V \cap C$  such that the number  $h = |\{i \in [d] : u_i = v'_i, u_i \neq *\}|$  is odd. Since  $V$  does not contain a twin pair, we have  $h \geq 3$ . Thus,  $d \geq 3$ . Let  $h = 3$  and  $\{i \in [d] : u_i = v'_i\} = \{i_1, i_2, i_3\}$ . For every  $j \in \{1, 2, 3\}$  let

$$x^j \in Eu_1 \times \cdots \times Eu_{i_j-1} \times Eu'_{i_j} \times Eu_{i_j+1} \times \cdots \times Eu_d.$$

The points  $x^1, x^2, x^3$  are pairwise different. Let us observe that for every  $k, m \in \{1, 2, 3\}, k \neq m$ , if  $x^k, x^m \in \check{w}$  for some  $w \in (*S)^d$ , then  $\check{u} \cap \check{w} \neq \emptyset$  and consequently  $w \notin V$ . Moreover,  $x^1, x^2, x^3 \notin \check{v}$  and  $x^1, x^2, x^3 \notin \check{u}$ . Therefore,  $|V| \geq 5$ . In the same manner we show that, if  $h \geq 5$ , then more than five words is needed to complete the set  $\{u, v\}$  to a partition code. Let  $|V| = 5$ , and let  $(v)_A = l_1 l_2 l_3$  and  $(u)_A = l'_1 l'_2 l'_3$ . By Lemma 2.2, we have  $\sum_{w \in V} \bar{g}(w, s_*) = 2^d$ . Suppose that  $\bar{g}(w, s_*) = 2^{d-1}$  for some  $w \in V \setminus \{v, u\}$ . Then there is exactly one  $i \in [d]$  such that  $w_i \neq *$  and  $(w)_i = * \cdots *$ . Since  $V$  is a partition code, it follows that  $(V \setminus \{w\})_i \subset (*S)^{d-1}$  is a partition code. This code does not contain a twin pair and consists of four words, which is impossible. Therefore, by Lemma 2.2, we have  $\bar{g}(u, s_*) = \bar{g}(v, s_*) = 2^{d-3}$  and  $\bar{g}(w, s_*) = 2^{d-2}$  for every  $w \in V \setminus \{v, u\}$ . Hence,  $(w)_{\{i_1, i_2, i_3\}} = * \cdots *$  for every  $w \in V$ , and since every two words in  $V$  are dichotomous, we have  $(V \setminus \{v, u\})_A = \{* l_2 l'_3, l'_1 * l_3, l_1 l'_2 *\}$  or  $(V \setminus \{v, u\})_A = \{* l'_2 l_3, l_1 * l'_3, l'_1 l_2 *\}$  (Figure 4).

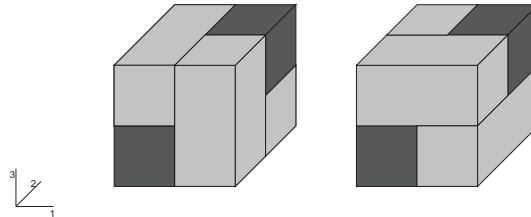


Fig. 4. Realizations of the codes  $(V)_A = \{l_1 l_2 l_3, l'_1 l'_2 l'_3, *l'_2 l_3, l_1 *l'_3, l'_1 l_2 *\}$  (on the left) and  $(V)_A = \{l_1 l_2 l_3, l'_1 l'_2 l'_3, *l_2 l'_3, l'_1 *l_3, l_1 l'_2 *\}$  (on the right).

*Proof of (b).* Let  $C, v, u$  and  $h$  be such as in the proof of the part (a). If  $h \geq 5$ , then, in the similar way as in the proof of (a), we show that more than five words is needed to complete the set  $\{u, v\}$  to a partition code. Thus,  $h = 3$  and  $\bar{g}(u, s_*) = \bar{g}(v, s_*) = 2^{d-3}$ . Let  $\{i \in [d] : u_i = v'_i, u_i \neq *\} = \{i_1, i_2, i_3\}$ . We use again Lemma 2.2. Let  $V = \{v^1, \dots, v^6\}$ ,  $v^3 = v, v^4 = u$ , and assume on the contrary that  $\bar{g}(v^i, s_*) \leq 2^{d-2}$  for every  $v^i \in V$ . The system of the equations

$$\sum_{i \geq 2} x_i 2^{d-i} = 2^d, \quad \sum_{i \geq 2} x_i = 6 \quad (3.2)$$

where  $x_3 \geq 2$  and  $x_i \in \{0, 1, \dots\}$  for  $i \in \{2, \dots, d\}$ , has only one solution:  $x_2 = 2, x_3 = 4$ . Let

$$\bar{g}(v^1, s_*) = \bar{g}(v^2, s_*) = 2^{d-2} \quad \text{and} \quad \bar{g}(v^i, s_*) = 2^{d-3} \quad (3.3)$$

for  $i = 3, \dots, 6$ . Let us consider the realization  $E(V)$ . We can assume that for every  $i \in [d]$  there is  $v \in V$  such that  $v_i \neq *$ ; otherwise, if  $v_i = *$  for some  $i \in [d]$  and every  $v \in V$ , we consider the code  $(V)_i$ . Every  $i$ -cylinder  $C_i = \bigcup \{\check{v}^j : v^j \in V, v_i^j \neq *\}$ ,  $i \in [d]$ , has to contain at least four boxes. Indeed, if  $C_i = \check{v}^j \cup \check{v}^k$ , then  $v^j$  and  $v^k$  are twins, and if  $C_i = \check{v}^j \cup \check{v}^k \cup \check{v}^n$ ,  $v_i^j = v_i^k, v_i^n = (v_i^k)'$ , then  $v^j$  and  $v^k$  are twins. From (3.3) it follows that we can always choose  $i \in [d]$  such that  $v_i^1 = v_i^2 = *$  or  $v_i^k = v_i^n = *$  for some  $k \in \{1, 2\}$  and  $n \in \{3, \dots, 6\}$ . In the first case we have  $C_i = \check{v}^3 \cup \check{v}^4 \cup \check{v}^5 \cup \check{v}^6$ . We can assume that  $v_i^3 = v_i^4, v_i^5 = v_i^6$  and  $v_i^3 = (v_i^5)'$ . By Lemma 3.2 the set  $\check{v}^3 \cup \check{v}^4$  is a  $L$ -polybox and thus, again by this lemma,  $\bar{g}(v^3, s_*) \neq \bar{g}(v^4, s_*)$ . This contradicts the assumption on  $v^3, v^4$ .

In the second case the set  $\bigcup \{\check{s} : s \in V \setminus \{v^k, v^n\}, s_i \neq *\}$  can not be an  $i$ -cylinder. Indeed, we have  $\bar{g}(v, s_*) = 2^{d-2}$  for exactly one  $v \in V \setminus \{v^k, v^n\}$  and  $\bar{g}(w, s_*) = 2^{d-3}$  for the rest three words  $w \in V \setminus \{v^k, v^n, v\}$ . Thus,  $|\check{v}| = (1/4)|ES|^d$  and  $|\check{w}| = (1/8)|ES|^d$  for the three words  $w \in V \setminus \{v^k, v^n, v\}$ . Therefore, the boxes from the set  $\{\check{s} : s \in V \setminus \{v^k, v^n\}, s_i \neq *\}$  can not be divided into two parts with the same sizes.

Thus, there is a word, say  $v^1$ , in the code  $V$  such that  $\bar{g}(v^1, s_*) = 2^{d-1}$ . If  $i \in [d]$  is such that  $v_i^1 \neq *$ , then  $v_i^2 = \dots = v_i^6 = (v_i^1)'$ , and hence  $V^{i, v_i^1} = \{v^1\}$  and  $V^{i, (v_i^1)'} = \{v^2, \dots, v^6\}$ . Since  $\{v^2, \dots, v^6\}$  does not contain a twin pair, by (a), we have  $d \geq 4$ . This completes the proof of the part (b).  $\square$

**COROLLARY 3.4** *Let  $V \subset S^d$  be a polybox code without twin pairs and let  $u \sqsubseteq V$  for some  $u \in S^d \setminus V$ . Then  $|V| \geq 5$ . If  $u = u_1 \dots u_d$  and  $|V| = 5$ , then*

$$(V)_A = \{l_1 l_2 l_3, l'_1 l'_2 l'_3, u_{i_1} l_2 l'_3, l'_1 u_{i_2} l_3, l_1 l'_2 u_{i_3}\}$$

or

$$(V)_A = \{l_1 l_2 l_3, l'_1 l'_2 l'_3, u_{i_1} l'_2 l_3, l_1 u_{i_2} l'_3, l'_1 l_2 u_{i_3}\},$$

where  $l_k \notin \{u_{i_k}, u'_{i_k}\}$  for  $k = 1, 2, 3$ ,  $A = [d] \setminus \{i_1, i_2, i_3\}$  and  $v_i = u_i$  for every  $i \in A$  and  $v \in V$ . In particular,  $V$  is rigid.

If  $|V| = 6$  and  $\check{u} \cap \check{v} \neq \emptyset$  for every  $v \in V$ , then  $d \geq 4$  and there are  $i \in [d]$  and  $v \in V$  such that  $(v)_i = (u)_i$ .

*Proof.* By Corollary 2.3, the set  $\bar{V}$  is a partition code. Since  $V$  does not contain a twin pair,  $\bar{V}$  does not contain a twin pair. By Lemma 3.3,  $|\bar{V}| \geq 5$ , and thus  $|V| \geq 5$ , because  $|\bar{V}| = |V|$ . Similarly, by Lemma 3.3, if  $|\bar{V}| = 5$ , it is of the form described in this lemma, and thus, by the definition of  $\bar{V}$ , the polybox code  $V$  has to be as predicted in the corollary. To show that  $V$  is rigid observe that if  $w \sqsubseteq V$ , then  $w_i = u_i$  for  $i \in A$ . If  $w_i \in \{l_1, l'_1, l_2, l'_2, l_3, l'_3\}$  for some  $i \in \{i_1, i_2, i_3\}$ , then the set  $\{\check{w} \cap \check{v} \neq \emptyset : v \in V\}$  contains at most three boxes. Since it is a suit for  $\check{w}$ , by the first part of the corollary and Lemma 2.1, there is a twin pair in  $V$ , which is a contradiction. Thus  $w_i \notin \{l_1, l'_1, l_2, l'_2, l_3, l'_3\}$  for every  $i \in \{i_1, i_2, i_3\}$ . If  $w_i \neq u_{i_j}$  for some  $j \in \{1, 2, 3\}$ , then  $g(v, w) = 2$  for at most two words  $v \in V$  and  $g(v, w) \leq 1$  for the rest  $v \in V$ . Then  $\sum_{v \in V} g(v, w) < 8$ , and by (2.4),  $w \not\sqsubseteq V$ . Consequently,  $V$  can cover only the word  $u$ , and thus it is rigid.

If  $|V| = 6$ , then  $|\bar{V}| = 6$  and, by Lemma 3.3 (b),  $d \geq 4$ . Moreover, there is  $i \in [d]$  and  $\bar{v} \in \bar{V}$  such that  $(\bar{v})_i = * \cdots *$ . By the definition of  $\bar{v}$  (given in Corollary 2.3) we have  $(v)_i = (u)_i$ .  $\square$

**COROLLARY 3.5** *Let  $V \subset S^d$  be a polybox code which does not contain a twin pair. If  $d \leq 3$ , then  $V$  is rigid.*

*Proof.* It is enough to prove the lemma for  $d = 3$ . If  $|V| \leq 5$ , then the rigidity of  $V$  is guaranteed by Corollary 3.4. Let  $|V| \geq 6$  and assume that  $w \sqsubseteq V$  for some  $w \in S^3$ . Let  $V_1 \subset V$  be such that  $w \sqsubseteq V_1$  and  $\check{w} \cap \check{v} \neq \emptyset$  for  $v \in V_1$ . By Corollary 2.3 and (2.2) the code  $V_1$  has to contain a pair of words  $v = v_1 v_2 v_3, u = u_1 u_2 u_3 \in V \cap C$  for some simple partition  $C \subset S^3$ , where the number  $h = |\{i : v_i = u'_i\}|$  is odd. Since  $V$  does not contain a twin pair,  $h = 3$ . If  $|V_1| = 5$ , then, by Corollary 3.4,  $V_1 = \{u_1 u_2 u_3, u'_1 u'_2 u'_3, w_1 u_2 u'_3, u'_1 w_2 u_3, u_1 u'_2 w_3\}$  or  $V_1 = \{u_1 u_2 u_3, u'_1 u'_2 u'_3, w_1 u'_2 u_3, u_1 w_2 u'_3, u'_1 u_2 w_3\}$ . Since the words in  $V$  are pairwise dichotomous, it is easy to check, that every word  $q \in V \setminus V_1$  has to form a twin pair with some word  $p \in V_1$ . The permissible forms of such words  $q$  are:  $l'_1 l'_2 l_3, l_1 l'_2 l'_3$  and  $l'_1 l_2 l'_3$ . If  $p = l'_1 l'_2 l_3$ , (we consider the first form of  $V_1$ ) then, by the dichotomy of the words,  $l_3 = u_3, l_2 = w'_2$  and  $l_1 = u'_1$ . Then  $q$  and  $u'_1 w_2 u_3$  are a twin pair.

If  $|V_1| = 6$ , then there is a twin pair in  $V_1$ , as we have  $d < 4$ . If  $|V_1| = 7$ , then, from [Theorem 5.1, [5]] it follows that, the polybox code  $V_1$  can be completed to a partition code  $V_1 \cup \{u\} \subset S^3$ . It is known that every partition code  $W \subset S^3$  contains at least three twin pairs. If  $W$  is layered, i.e.  $W = W^{i,l} \cup W^{i,l'}$  for some  $i \in [3]$  and  $l \in S$  (see [16]), then there are two twin pairs in each of the codes  $W^{i,l}$  and  $W^{i,l'}$ , as they are partition codes in two dimensional case. If  $W \neq W^{i,l} \cup W^{i,l'}$  for every  $i \in [3]$ , then  $W$  has the structure as presented in Figure 1. If  $V_1 \cup \{u\}$  contains three twin pairs:  $u, v^1$ ;  $u, v^2$  and  $u, v^3$ , where  $v^1, v^2, v^3 \in V_1$ , then it must be layered, and thus there is  $v^4 \in V_1$  which forms a twin pair with  $v^1, v^2$  or  $v^3$ . If it is not layered, then  $V_1$  contains a twin pair. Thus, if  $V$  do not contain a twin pair, then it is rigid.  $\square$

For fixed  $x \in ES$  and  $i \in [d]$  let

$$\pi_x^i = ES \times \cdots \times ES \times \{x\} \times ES \times \cdots \times ES,$$

where  $\{x\}$  stands at the  $i$ th position. If  $V \subset (*S)^d$  is a polybox code, then the slice  $\pi_x^i \cap \bigcup E(V)$  is a "flat" polybox in  $(ES)^d$  (boxes which are contained in

this polybox have the factor  $\{x\}$  at the  $i$ th position). Therefore we define a polybox  $(\pi_x^i \cap \bigcup E(V))_i$  in the  $(d-1)$ -box  $(ES)^{d-1}$ :

$$(\pi_x^i \cap \bigcup E(V))_i = \bigcup \{(\check{v})_i : v \in V \text{ and } \pi_x^i \cap \check{v} \neq \emptyset\}.$$

The polybox  $(\pi_x^i \cap \bigcup E(V))_i$  does not depend on a particular choice of a polybox code, because if  $W$  is an equivalent polybox code to  $V$ , then  $\bigcup E(V) = \bigcup E(W)$ , and hence  $(\pi_x^i \cap \bigcup E(V))_i = (\pi_x^i \cap \bigcup E(W))_i$ .

One of the main proof technique which is used in this paper derives from geometric tomography. We will slice given polybox  $\bigcup E(V)$  by the set  $\pi_x^i$  for various  $x \in ES$ . The information collected on slices  $\pi_x^i \cap \bigcup E(V)$  will enable us to find the structure of  $V$ .

Recall that, the box number  $|F|_0$  is the number of boxes in any proper suit for the polybox  $F$ . From (2.1) we deduce the following useful lemma.

**LEMMA 3.6** *Let  $V \subseteq S^d$  be a polybox code, and let  $E(V)$  be the equicomplementary realization of  $V$ . Assume that there are letters  $l_1, l_2 \in S$  and there is  $i \in [d]$  such that  $|(\pi_x^i \cap \bigcup E(V))_i|_0 \geq m$  for every  $x \in El_1 \cap El_2$  and  $|(\pi_y^i \cap \bigcup E(V))_i|_0 \geq n$  for every  $y \in El'_1 \cap El'_2$ . Then  $|V| \geq m + n$ .*

*Proof.* Let  $A \subseteq S$  be such that  $V = \bigcup_{p \in A} V^{i,p}$  and  $V^{i,p} \neq \emptyset$  for every  $p \in A$ . Let us divide the set  $S$  into two nonempty and disjoint sets  $S^0$  and  $S^1$  such that  $S^1 = \{s' : s \in S^0\}$ . We also divide the set  $A$  into two disjoint sets  $B$  and  $C$  such that  $B \subseteq S^0$  and  $C \subseteq S^1$ . Let  $B_1 = B \cup C'$  and  $C_1 = C \cup B'$ , where  $B' = \{l' : l \in B\}$  and  $C' = \{l' : l \in C\}$ . Then, by (2.1), the sets  $\bigcap_{p \in B_1} Ep \cap El_1 \cap El_2$  and  $\bigcap_{p \in C_1} Ep \cap El'_1 \cap El'_2$  are nonempty. Let  $x \in \bigcap_{p \in B_1} Ep \cap El_1 \cap El_2$  and  $y \in \bigcap_{p \in C_1} Ep \cap El'_1 \cap El'_2$ . Then  $|(\pi_x^i \cap \bigcup E(V))_i|_0 = \sum_{p \in B} |V^{i,p}| \geq m$  and  $|(\pi_y^i \cap \bigcup E(V))_i|_0 = \sum_{p \in C} |V^{i,p}| \geq n$ . Since  $|V| = \sum_{p \in B} |V^{i,p}| + \sum_{p \in C} |V^{i,p}|$ , it follows that  $|V| \geq n + m$ .  $\square$

**LEMMA 3.7** *Let  $V \subset S^d$  be a polybox code,  $w \in S^d$  and  $w \sqsubseteq V$ . Assume that there is  $i \in [d]$  such that  $0 < |\{\check{w} \cap \check{v} \neq \emptyset : v \in V, v_i \notin \{w_i, w'_i\}\}| \leq 3$ . Then there is a twin pair in  $V$ . In particular, if  $\check{w} \cap \check{v} \neq \emptyset$  for every  $v \in V$  and  $(|V| - 3) \leq |V^{i,w_i}| < |V|$ , then there is a twin pair in  $V$ .*

*Proof.* Since  $\{\check{w} \cap \check{v} \neq \emptyset : v \in V\}$  is a suit for the  $d$ -box  $\check{w}$ , the set  $C_i = \bigcup \{\check{w} \cap \check{v} \neq \emptyset : v \in V, v_i \notin \{w_i, w'_i\}\}$  is an  $i$ -cylinder in the  $d$ -box  $\check{w}$ . By the assumption there are at most three boxes in  $C_i$ . Assume that  $C_i = \check{w} \cap \check{u} \cup \check{w} \cap \check{v}$ . Then  $u_i = v'_i$ . Clearly,  $\check{w} \cap \check{u}, \check{w} \cap \check{v}$  form a twin pair. Thus, by Lemma 2.1, the words  $u, v \in V$  are a twin pair. If  $C_i = \check{w} \cap \check{u} \cup \check{w} \cap \check{v} \cup \check{w} \cap \check{q}$ , then assuming that  $u_i = v'_i$  and  $u_i = q'_i$ , we have  $(\check{w} \cap \check{u})_i = (\check{w} \cap \check{v})_i \cup (\check{w} \cap \check{q})_i$ , and then  $(\check{w} \cap \check{v})_i$  and  $(\check{w} \cap \check{q})_i$  are a twin pair and thus  $\check{w} \cap \check{v}$  and  $\check{w} \cap \check{q}$  are a twin pair. By Lemma 2.1, the words  $v$  and  $q$  are a twin pair.

Now we list the special cases in which Theorem 3.1 holds.

**STATEMENT 3.8** *Let  $V, W \subset S^d$  be two equivalent polybox codes without twin pairs, and let  $V \cap W = \emptyset$ .*

(a) *If there are words  $v \in V$  and  $w \in W$  such that  $(v)_i = (w)_i$  for some  $i \in [d]$ , then  $|V| \geq 12$ .*

- (b) If there are  $i \in [d]$  and  $l \in S$  such that  $|V^{i,l}| \geq 5$  and  $|V^{i,l'}| \geq 5$ , then  $|V| \geq 12$ .
- (c) If there are  $i \in [d]$  and  $l \in S$  such that  $V^{i,l} \neq \emptyset$ ,  $V^{i,l'} \neq \emptyset$  and  $|V^{i,l} \cup V^{i,l'}| \leq 3$ , then  $|V| \geq 12$ .
- (d) If there are  $i \in [d]$  and  $l \in S$  such that  $V^{i,l} \neq \emptyset$  and  $V^{i,l'} = \emptyset$ , then  $|V| \geq 12$ .
- (e) If there are  $i \in [d]$  and pairwise different and pairwise non-dichotomous letters  $l, s, p \in S$  such that  $V^{i,r} \cup V^{i,r'} \neq \emptyset$  for every  $r \in \{l, s, p\}$ , then  $|V| \geq 12$ .
- (f) If there are  $i \in [d]$  and  $l \in S$  such that  $|V^{i,l}| = 1$ , then  $|V| \geq 12$ .

*Proof.* We assume that  $d$  is the smallest number for which polybox codes  $V, W \subset S^d$  exist, and the number  $|V|$  is minimal in the sense that if  $\bar{V} \subset S^d$  is a polybox code without twin pairs,  $\bar{W} \subset S^d$  is an equivalent code to  $\bar{V}$  and  $|\bar{V}| < |V|$ , then  $\bar{V} = \bar{W}$  or  $\bar{W}$  contains a twin pair.

*Proof of (a)* Assume, for simplicity of notation, that  $i = d$ ,  $v = a \cdots a$  and  $w = a \cdots ab$ ,  $a \neq b$ . By (2.1), the sets  $\check{w} \setminus \check{v}$  and  $\check{v} \setminus \check{w}$  are nonempty. Since  $(w)_d = (v)_d$ , we have  $\check{w} \setminus \check{v} \subset \bigcup E(V^{d,a'})$  and consequently  $(\check{w})_d \subseteq \bigcup (E(V^{d,a'}))_d$ . Thus, by (2.4),  $(w)_d \subseteq (V^{d,a'})_d$ . (Clearly,  $(w)_d \notin (V^{d,a'})_d$ ; otherwise there is a twin pair in  $V^{d,a} \cup V^{d,a'}$ , which is impossible). Then, by Corollary 3.4,  $|V^{d,a'}| \geq 5$ . In the same manner we show that  $|W^{d,b'}| \geq 5$ . Let us assume that  $|V^{d,a'}| = 5$ . Then, by Corollary 3.4, the code  $V^{d,a'}$  is rigid, and thus the code  $(V^{d,a'})_d \subset S^{d-1}$  is rigid. If

$$\pi_x^d \cap \bigcup E(V) = \pi_x^d \cap \bigcup E(V^{d,a'})$$

for some  $x \in Ea' \cap Eb$ , then  $(w)_d \in (V^{d,a'})_d$ , because  $\pi_x^d \cap \bigcup E(W) = \pi_x^d \cap \bigcup E(V)$  and  $(V^{d,a'})_d$  is rigid. But then the word  $u = a \cdots aa'$  belongs to  $V^{d,a'}$  and forms a twin pair with the word  $v$ , which contradicts the assumption. Therefore,  $|V^{d,a'}| \geq 6$  or  $\pi_x^d \cap \bigcup E(V) \neq \pi_x^d \cap \bigcup E(V^{d,a'})$  for every  $x \in Ea' \cap Eb$ . Then

$$|(\pi_x^d \cap \bigcup E(V))_d|_0 \geq 6$$

for every  $x \in Ea' \cap Eb$ . In the same manner we show that

$$|(\pi_y^d \cap \bigcup E(W))_d|_0 \geq 6$$

for every  $y \in Ea \cap Eb'$ . Clearly,  $(\pi_x^d \cap \bigcup E(W))_d = (\pi_x^d \cap \bigcup E(V))_d$  for every  $x \in ES$ . By Lemma 3.6, we have  $|V| \geq 12$ .

*Proof of (b).* Let us assume that  $|V^{i,l}| = 5$ , and suppose that there is  $x \in El$  such that

$$\pi_x^i \cap \bigcup E(V) = \pi_x^i \cap \bigcup E(V^{i,l}).$$

Since  $\pi_x^i \cap \bigcup E(V) = \pi_x^i \cap \bigcup E(W)$  and, by Corollary 3.4, the polybox  $(\pi_x^i \cap \bigcup E(V^{i,l}))_i$  is rigid, there are words  $v \in V^{i,l}$  and  $w \in W$  such that  $(v)_i = (w)_i$ . Then, by (a), we have  $|V| \geq 12$ . Thus, we assume that

$$|V^{i,l}| \geq 6 \text{ or } \pi_x^d \cap \bigcup E(V) \neq \pi_x^i \cap \bigcup E(V^{i,l})$$

for every  $x \in El$  and

$$|V^{i,l'}| \geq 6 \text{ or } \pi_y^d \cap \bigcup E(V) \neq \pi_y^i \cap \bigcup E(V^{i,l'})$$

for every  $y \in El'$ . Then, for every  $x \in El$  and  $y \in El'$ , we have

$$|(\pi_x^d \cap \bigcup E(V))_d|_0 \geq 6 \quad \text{and} \quad |(\pi_y^d \cap \bigcup E(V))_d|_0 \geq 6.$$

From Lemma 3.6, in which we take  $l_1 = l_2 = l$ , it follows that  $|V| \geq 12$ .

*Proof of (c).* Observe that,  $V^{i,l} \subseteq W^{i,l}$  and  $V^{i,l'} \subseteq W^{i,l'}$ . To justify this, suppose on the contrary that  $V^{i,l} \not\subseteq W^{i,l}$ . By (2.4), there are words  $v \in V^{i,l}$  and  $w \in W \setminus (W^{i,l} \cup W^{i,l'})$  such that  $\check{w} \cap \check{v} \neq \emptyset$ . Since  $w \subseteq V$ ,  $l \notin \{w_i, w'_i\}$  and  $0 < |\{\check{w} \cap \check{v} \neq \emptyset : v \in V^{i,l} \cup V^{i,l'}\}| \leq 3$ , by Lemma 3.7, there is a twin pair in  $V$ , which is a contradiction.

Therefore,  $V^{i,l} \subseteq W^{i,l}$  and  $V^{i,l'} \subseteq W^{i,l'}$ . By Corollary 3.4,  $|W^{i,l}| \geq 5$  and  $|W^{i,l'}| \geq 5$ , and from (b) we get  $|V| \geq 12$ .

*Proof of (d).* Since  $V^{i,l'} = \emptyset$ , it follows that  $V^{i,l} \subseteq W^{i,l}$ , and thus, by Corollary 3.4, we have  $|W^{i,l}| \geq 5$ . Let us suppose that  $|W^{i,l}| = 5$  and

$$\pi_x^i \cap \bigcup E(W) = \pi_x^i \cap \bigcup E(W^{i,l})$$

for some  $x \in El$ . Since  $(\pi_x^i \cap \bigcup E(V))_i = (\pi_x^i \cap \bigcup E(W))_i$  and the code  $(W^{i,l})_i$  is, by Corollary 3.4, rigid, there is  $v \in V$  such that  $(v)_i \in (W^{i,l})_i$ . By (a), we have  $|V| \geq 12$ . So we assume that

$$|(\pi_x^i \cap \bigcup E(W))_i|_0 \geq 6, \tag{3.4}$$

for every  $x \in El$ . From the assumption on the number  $d$  it follows that the codes  $V^{i,l}$  and  $W^{i,l}$  are not equivalent. Therefore, by (2.4), there are words  $w \in W^{i,l}$  and  $v \in V^{i,s}$ , where  $s \notin \{l, l'\}$  such that  $\check{w} \cap \check{v} \neq \emptyset$ . Hence,  $W^{i,l'} \neq \emptyset$  and  $V^{i,s'} \neq \emptyset$ . Let  $u \in W^{i,l'}$ . The set of boxes  $\{\check{u} \cap \check{v} \neq \emptyset : v \in V\}$  is a suit for the  $d$ -box  $\check{u}$ . Assume first, that  $u \subseteq V^{i,s} \cup V^{i,s'}$ . (Notice that, since  $l \notin \{s, s'\}$ , we have  $\check{u} \cap \bigcup E(V^{i,s}) \neq \emptyset$  and  $\check{u} \cap \bigcup E(V^{i,s'}) \neq \emptyset$ ). If  $(u)_i \in (V^{i,s})_i$  or  $(u)_i \in (V^{i,s'})_i$ , then, by (a),  $|V| \geq 12$ . Thus, we can assume that  $(u)_i \notin (V^{i,s})_i$  and  $(u)_i \notin (V^{i,s'})_i$ . Since  $(u)_i \subseteq (V^{i,s})_i$  and  $(u)_i \subseteq (V^{i,s'})_i$ , we obtain, by Corollary 3.4,  $|(V^{i,s})_i| \geq 5$  and  $|(V^{i,s'})_i| \geq 5$ . From (b) we get  $|V| \geq 12$ .

Now let us assume that there are two letters  $s, p \in S \setminus \{l, l'\}$ ,  $s \notin \{p, p'\}$ , such that

$$u \subseteq V^{i,s} \cup V^{i,s'} \cup V^{i,p} \cup V^{i,p'}$$

and  $\check{u} \cap \bigcup E(V^{i,r}) \neq \emptyset$  for every  $r \in \{s, s', p, p'\}$ . If  $(V^{i,r'})_i \subseteq (V^{i,r})_i$  for some  $r \in \{s, s', p, p'\}$ , then  $|(V^{i,r})_i| \geq 5$ , and thus  $|V^{i,r}| \geq 5$ . The sets  $V^{i,t}$  and  $V^{i,t'}$ , where  $t \in \{s, p\} \setminus \{r\}$ , are nonempty, and therefore

$$|(\pi_x^i \cap \bigcup E(V))_i|_0 \geq 6,$$

for every  $x \in Er$ . Combining this with (3.4) (we have  $\pi_x^i \cap \bigcup E(V) = \pi_x^i \cap \bigcup E(W)$ ) and using Lemma 3.6, in which we take  $l_1 = l$  and  $l_2 = r'$ , we get  $|V| \geq 12$ . Therefore we may assume that  $(V^{i,r'})_i \not\subseteq (V^{i,r})_i$  for every  $r \in \{s, s', p, p'\}$ . Then, by (2.4), we have  $\bigcup E((V^{i,r'})_i) \setminus \bigcup E((V^{i,r})_i) \neq \emptyset$  for every  $r \in \{s, s', p, p'\}$ . Thus,  $W^{i,r} \neq \emptyset$  for every  $r \in \{s, s', p, p'\}$ . If  $|W^{i,r} \cup W^{i,r'}| \leq 3$  for some  $r \in \{s, p\}$  then, by (c), we get  $|V| \geq 12$ . If  $|W^{i,r} \cup W^{i,r'}| \geq 4$  for every  $r \in \{s, p\}$ , then  $|V| = |W| \geq |W^{i,s} \cup W^{i,s'}| + |W^{i,p} \cup W^{i,p'}| + |W^{i,l}| > 12$ .

*Proof of (e).* If  $V^{i,r} \cup V^{i,r'} \neq \emptyset$  for every  $r \in \{l, s, p\}$ , then, by (d) and (c), we may assume that  $V^{i,r} \neq \emptyset$  for every  $r \in \{l, l', s, s', p, p'\}$  and  $|V^{i,r} \cup V^{i,r'}| \geq 4$  for every  $r \in \{l, s, p\}$ . Then  $|V| \geq 12$ .

*Proof of (f).* Before proving this statement, let us note that if  $V = V^{i,l} \cup V^{i,l'}$ ,  $V^{i,l} \neq \emptyset$ ,  $V^{i,l'} \neq \emptyset$ , for some  $i \in [d]$  and  $l \in S$ , then  $W = W^{i,l} \cup W^{i,l'}$  or there is at least one  $s \in S \setminus \{l, l'\}$  such that the codes  $(W^{i,s})_i$  and  $(W^{i,s'})_i$  are nonempty and equivalent. But then  $V \cap W \neq \emptyset$  or  $W$  contains a twin pair. To verify this, observe that if  $W = W^{i,l} \cup W^{i,l'}$ , then the codes  $V^{i,l}$  and  $W^{i,l}$  are equivalent, and then, by the assumption on the number  $d$ , we have  $V^{i,l} = W^{i,l}$ . Let now  $s \in S \setminus \{l, l'\}$  be such that  $W^{i,s} \neq \emptyset$ . If  $(W^{i,s})_i$  and  $(W^{i,s'})_i$  are not equivalent, then we can assume that  $\bigcup E(W^{i,s}) \setminus \bigcup E(W^{i,s'}) \neq \emptyset$ . Then  $\check{v} \cap \bigcup E(W^{i,s}) \setminus \bigcup E(W^{i,s'}) \neq \emptyset$  for some  $v \in V$ , and consequently  $V^{i,s} \neq \emptyset$ , a contradiction. Therefore,  $(W^{i,s})_i$  and  $(W^{i,s'})_i$  are equivalent, and thus, by the assumption on the number  $d$ ,  $(W^{i,s})_i = (W^{i,s'})_i$ , which means that there are twin pairs in  $W^{i,s} \cup W^{i,s'}$ . Therefore in what follows we assume (also by (d)) that for every  $i \in [d]$  there are at least two letters  $l_1, l_2 \in S$ ,  $l_1 \notin \{l_2, l'_2\}$  such that  $V^{i,r} \neq \emptyset$  for every  $r \in \{l_1, l'_1, l_2, l'_2\}$ .

Now we prove (f).

Let  $V^{i,l} = \{u\}$ . Assume first that  $V^{i,l} \not\subseteq W^{i,l}$ . Then there is  $w \in W \setminus (W^{i,l} \cup W^{i,l'})$  such that  $\check{w} \cap \check{u} \neq \emptyset$ , and thus  $\check{w} \cap \bigcup E(V^{i,l'}) \neq \emptyset$  (and then  $V^{i,l'} \not\subseteq W^{i,l'}$ ). Since  $w \subseteq V$ , the set  $\check{w} \cap \check{u} \cup \check{w} \cap \bigcup E(V^{i,l'})$  is an  $i$ -cylinder in the  $d$ -box  $\check{w}$ . Therefore,  $(\check{w} \cap \check{u})_i = \bigcup \{(\check{w} \cap \check{v})_i \neq \emptyset : v \in V^{i,l'}\}$ , and thus

$$Ew_j \cap Ev_j \subseteq Ew_j \cap Eu_j \quad (3.5)$$

for every  $j \in [d] \setminus \{i\}$  and every  $v = v_1 \cdots v_d \in V^{i,l'}$  for which  $(\check{w} \cap \check{v})_i \neq \emptyset$ . If  $w_j = u_j$  for every  $j \in [d] \setminus \{i\}$ , then  $(w)_i = (u)_i$  and, by (a), we have  $|V| \geq 12$ . Let  $k \in [d] \setminus \{i\}$  be such that  $w_k \neq u_k$ . Then  $v_k = u_k$  for every  $v \in V^{i,l'}$  with  $(\check{w} \cap \check{v})_i \neq \emptyset$ , as if  $v_k \neq u_k$  for some  $v$ , then, by (2.1),  $Ew_k \cap Ev_k \not\subseteq Ew_k \cap Eu_k$ , which contradicts (3.5). Therefore,  $v \in V^{k,u_k}$  for every  $v \in V^{i,l'}$  such that  $(\check{w} \cap \check{v})_i \neq \emptyset$ . Since the set of boxes  $\{(\check{w} \cap \check{v})_i \neq \emptyset : v \in V^{i,l'}\}$  is a suit for the box  $(\check{w} \cap \check{u})_i$ , which, by (2.1) and the fact that  $V$  is twin pairs free, does not contain a twin pair, it has to contain, by Corollary 3.4, at least five elements. But  $u \in V^{k,u_k}$ , and thus  $|V^{k,u_k}| \geq 6$ . If  $|V^{k,u'_k}| \geq 2$ , then assuming, by (c), that  $|V^{k,l} \cup V^{k,l'}| \geq 4$ , where  $l \notin \{u_k, u'_k\}$ , we get  $|V| \geq 12$ . So, by (d), we assume that  $V^{k,u'_k} = \{p\}$  for some  $p = p_1 \cdots p_d \in S^d$ . Clearly,  $\check{w} \cap \check{p} \neq \emptyset$ , as  $w_k \notin \{u_k, u'_k\}$ , and  $\bigcup \{(\check{w} \cap \check{v})_k \neq \emptyset : v \in V^{k,u_k}\} = (\check{w} \cap \check{p})_k$ . Then, in the similar way as above, we show that  $(w)_k = (p)_k$ , and then, by (a),  $|V| \geq 12$  or  $|V^{m,p_m}| \geq 7$  for some  $m \in [d]$ . This inequality follows from that fact that now the set  $\{(\check{w} \cap \check{v})_k \neq \emptyset : v \in V^{k,u_k}\}$  contains at least six boxes. Using the same arguments as before we show that there is  $m \in [d]$  such that  $v_m = p_m$  for every  $v = v_1 \cdots v_d \in V^{k,u_k}$  such that  $(\check{w} \cap \check{v})_k \neq \emptyset$ , which gives  $|V^{m,p_m}| \geq 6$ . But  $p \in V^{m,p_m}$  and  $p \notin V^{k,u_k}$ . Thus,  $|V^{m,p_m}| \geq 7$ . By (d) and (c), we can assume that  $|V^{m,p'_m}| \geq 1$  and  $|V^{m,l} \cup V^{m,l'}| \geq 4$  for some  $l \in S \setminus \{p_m, p'_m\}$ . Then  $|V| \geq 12$ .

Now assume that  $V^{i,l} \subseteq W^{i,l}$  and  $V^{i,l'} \subseteq W^{i,l'}$ . By (d), we can assume that  $V^{i,l} \neq \emptyset$  and  $V^{i,l'} \neq \emptyset$ , and then, by Corollary 3.4, we have  $|W^{i,l}| \geq 5$  and  $|W^{i,l'}| \geq 5$ . Thus, by (b), we get  $|V| \geq 12$ .  $\square$

In the proof of Theorem 3.1 we will use notions from graph theory.



A pair of words  $v, u \in S^d$  such that  $v_i \notin \{u_i, u'_i\}$  for some  $i \in [d]$  and  $(u)_i$  and  $(v)_i$  are a twin pair is called an *i-siblings*.

Let  $V \subset S^d$  be a polybox code without twin pairs. We define a graph  $G = (V, \mathcal{E})$  in which two vertices  $u, v \in V$  are adjacent if they are *i-siblings* for some  $i \in [d]$ . Moreover, we colour each edge in  $\mathcal{E}$  with the colours from the set  $[d]$ : an edge  $e \in \mathcal{E}$  has a colour  $i \in [d]$  if its ends are *i-siblings*. The graph  $G$  is simple and since the code  $V$  does not contain twin pairs, we have  $d(v) \leq d$  for every  $v \in V$ , where  $d(v)$  denotes the number of neighbors of  $v$ . Observe that the graph  $G$  does not contain triangles.

Let vertices  $u$  and  $v$  be adjacent, and let  $d(u) = n$  and  $d(v) = m$ . It can be easily show that if  $n + m = 2d$ , then there are  $i \in [d]$  and  $l \in S$  such that

$$|V^{i,l} \cup V^{i,l'}| \geq n + m - 2, \quad (3.6)$$

and if  $n + m \leq 2d - 1$ , then

$$|V^{i,l} \cup V^{i,l'}| \geq n + m - 1 \quad (3.7)$$

for some  $i \in [d]$  and  $l \in S$ .

By  $d(G)$  we denote the average degree of a graph  $G$ , and  $N(S)$  denotes the set of all neighbors of vertices  $v \in S$ . In the sequel we will need the following two lemmas.

**LEMMA 3.9** *Let  $G = (V, \mathcal{E})$  be a simple graph, and let  $m = \max\{d(v) + d(u) : v, u \in V \text{ and } v, u \text{ are adjacent}\}$ . Then  $d(G) \leq m/2$ .*

*Proof.* Let  $V_1 \subset V$  be the set of all vertices  $v$  such that  $d(v) > m/2$ , and let  $\mathcal{E}_1 \subset \mathcal{E}$  be the set of all edges which are incident with vertices from  $V_1$ . Since there is no edge with ends in the set  $V_1$ , the graph  $G_1 = (V, \mathcal{E}_1)$  is a bipartite with the bipartition  $\{V_1, V \setminus V_1\}$ . We will show that the graph  $G_1$  contains a matching of  $V_1$ . To do this, let  $S \subset V_1$ . The number of edges in  $\mathcal{E}_1$  which are incident with vertices from  $S$  is greater than  $|S|m/2$ . On the other hand the number of edges in  $\mathcal{E}_1$ , which are incident with vertices from  $N(S) \subset V \setminus V_1$ , is at most  $|N(S)|m/2$ . Each edge from  $\mathcal{E}_1$  is incident with  $S$  if and only if it is incident with  $N(S)$ . Therefore,  $|N(S)|m/2 > |S|m/2$ , and thus  $|N(S)| > |S|$ . By the marriage theorem, there is a matching of the set  $V_1$ . Let  $V_2 \subset V \setminus V_1$  be the set of endpoints of edges from the matching of  $V_1$ . Then  $|V_1| = |V_2|$  and consequently

$$\begin{aligned} d(G) &= \frac{\sum_{v \in V_1} d(v) + \sum_{v \in V_2} d(v) + \sum_{v \in V \setminus (V_1 \cup V_2)} d(v)}{|V|} \\ &\leq \frac{|V_1|m + (|V| - 2|V_1|)\frac{m}{2}}{|V|} = \frac{m}{2}. \end{aligned}$$

□

**LEMMA 3.10** *Let  $V \subset S = \{a, a', b, b'\}^d$  be a polybox code without twin pairs. If  $|V| \leq 7$ , then  $V$  is rigid.*

*Proof.* We will show that if  $W$  is an equivalent polybox code to  $V$  and  $V \cap W = \emptyset$ , then  $|V| > 7$ . We proceed by induction on  $d$ . By Corollary 3.5, the lemma is

true for  $d \leq 3$ . Let  $d \geq 4$ . We can assume that for every  $i \in [d]$  there is at least one letter  $l \in S$  such that  $V^{i,l} \neq \emptyset$  and  $V^{i,l'} \neq \emptyset$ , as if  $V^{i,l} \neq \emptyset$ ,  $V^{i,s} \neq \emptyset$  and  $V^{i,l'} = \emptyset$ ,  $V^{i,s'} = \emptyset$ , where  $l \notin \{s, s'\}$  then, by the inductive hypothesis,  $V^{i,l} = W^{i,l}$  and  $V^{i,s} = W^{i,s}$ , and thus  $V \cap W \neq \emptyset$ .

Assume that  $V^{i,a} \neq \emptyset$ ,  $V^{i,a'} \neq \emptyset$  and  $V^{i,b} \cup V^{i,b'} = \emptyset$  for some  $i \in [d]$ . Then, again by the inductive hypothesis, for every  $x \in ES$  the polybox  $(\pi_x^i \cap \bigcup E(V^{i,l}))_i$  is rigid for  $l \in \{a, a'\}$ . Thus, there are  $u \in V^{i,a}$ ,  $v \in V^{i,a'}$  and  $w \in W^{i,b}$ ,  $q \in W^{i,b'}$  such that  $(u)_i = (w)_i$  and  $(v)_i = (q)_i$ . This implies  $(u)_i \subseteq (V^{i,a'})_i$  and  $(v)_i \subseteq (V^{i,a})_i$ . By Corollary 3.4 and the fact that  $V$  is twin pairs free,  $|V| = |(V^{i,a})_i| + |(V^{i,a'})_i| \geq 10$ .

Now assume that  $V^{i,a} \neq \emptyset$ ,  $V^{i,a'} \neq \emptyset$ ,  $V^{i,b} \neq \emptyset$  and  $V^{i,b'} = \emptyset$  for some  $i \in [d]$ . Take  $x \in Ea \cap Eb'$ . Clearly, the polybox  $(\pi_x^i \cap \bigcup E(V^{i,a}))_i$  is rigid. Therefore there is  $w \in W^{i,b'}$  such that  $(u)_i = (w)_i$  for some  $u \in V^{i,a}$ . In the same way as above we conclude that  $|V^{i,a'}| \geq 5$ . Assume that  $|V^{i,a'}| = 5$ ,  $|V^{i,a}| = 1$ ,  $|V^{i,b}| = 1$ . If now we choose  $x \in Ea' \cap Eb'$ , then, by Corollary 3.4, the polybox  $(\pi_x^i \cap \bigcup E(V^{i,a'}))_i$  is rigid. Hence, for every  $u \in V^{i,a'}$  there is  $w \in W \setminus W^{i,a} \cup W^{i,a'}$  such that  $(u)_i = (w)_i$ . Then  $u \subseteq V^{i,a}$ , and consequently  $|V^{i,a}| \geq 5$ , a contradiction.

Therefore we assume that  $V^{i,l} \neq \emptyset$  for every  $i \in [d]$  and  $l \in S$ .

Let us suppose that there are  $i \in [d]$  and two letters in  $S$ , say  $a$  and  $b$ , such that the polybox code  $(V^{i,a} \cup V^{i,b})_i$  does not contain a twin pair. This means, by the inductive hypothesis, that for  $x \in Ea \cap Eb$  the polybox  $(\pi_x^i \cap \bigcup E(V))_i = (\pi_x^i \cap \bigcup E(V^{i,a} \cup V^{i,b}))_i$  is rigid. Then for fixed  $u \in V^{i,a}$  and  $v \in V^{i,b}$  there are  $w \in W^{i,b}$  and  $q \in W^{i,a}$  such that  $(u)_i = (w)_i$  and  $(v)_i = (q)_i$ . Thus,  $(u)_i \subseteq (V^{i,a'})_i$  and  $(v)_i \subseteq (V^{i,b'})_i$ . As  $V$  does not contain twin pairs, by Corollary 3.4, we have  $|V| \geq |(V^{i,a'})_i| + |(V^{i,b'})_i| \geq 10$ .

Therefore we can assume that for every  $i \in [d]$  and every two letters  $l, s \in S$ ,  $l \notin \{s, s'\}$ , there are  $i$ -siblings in the set  $V^{i,l} \cup V^{i,s}$ . Then for every  $i \in [d]$  there are at least 4 edges with the colour  $i$  in the graph  $G = (V, \mathcal{E})$  and consequently there are at least  $4d$  edges in the set  $\mathcal{E}$ . Let  $u^0, v^0 \in V$  be such that

$$d(v^0) + d(u^0) = \max\{d(v) + d(u) : v, u \in V \text{ and } v, u \text{ are adjacent}\}.$$

Since  $|V| \leq 7$ , we have  $d(v^0) + d(u^0) \leq 7$ , and then from Lemma 3.9 it follows that

$$d(G) \leq \frac{7}{2}.$$

But  $d(G)|V| = 2|\mathcal{E}|$  and  $2|\mathcal{E}| \geq 32$ . Therefore,  $|V| > 7$ .  $\square$

Now we can prove Theorem 3.1.

*The proof of Theorem 3.1.* By Corollary 3.5 and Statement 3.8 we make the following assumptions:

(A):  $d \geq 4$ ,  $S = \{a, a', b, b'\}$  and  $|V^{i,l}| \geq 2$  for every  $i \in [d]$  and  $l \in S$ .

For every  $w \in W$  let  $V_w \subseteq V$  be such that  $w \subseteq V_w$  and  $\check{w} \cap \check{v} \neq \emptyset$  for all  $v \in V_w$ . If  $|V_w| < 5$  for some  $w \in W$ , then, by Corollary 3.4, there is a twin pair in  $V_w$ . If  $|V_w| = 6$  for some  $w \in W$ , then, by Corollary 3.4, there is  $v \in V$  and  $i \in [d]$  such that  $(v)_i = (w)_i$ . Thus, by Statement 3.8 (a),  $|V| \geq 12$ . Finally, if  $10 \leq |V_w| \leq 11$  for some  $w \in W$ , then  $|V| \geq 12$  as  $V_w \cap V^{i,w'_i} = \emptyset$  for every  $i \in [d]$  and, by (A),  $|V^{i,w'_i}| \geq 2$ . Therefore in what follows we assume that  $|V_w| \in \{5, 7, 8, 9\}$ .

First we prove the theorem for  $d = 4$ .

Since  $v \sqsubseteq W$  for  $v \in V$  and  $W$  does not contain twin pairs, from Corollary 2.3 and (2.2) it follows that there are  $w^1, w^2 \in W$  such that  $|\{i : w_i^1 = (w_i^2)'\}| = 3$  and  $w_j^1 = w_j^2$  for  $j \notin \{i : w_i^1 = (w_i^2)'\}$ . For simplicity of notation we assume that  $w^1 = bbbb$  and  $w^2 = b'b'b'b$ . By Statement 3.8 (a) we can assume that  $g(v, w) \leq 4$  for every  $v \in V_w$  and  $w \in W$ . Thus,  $g(v, w^1), g(v, w^2) \in \{1, 2, 4\}$  for  $v \in V$ .

For  $k \in \{7, 8, 9\}$  the system of equations

$$4x + 2y + z = 16, \quad x + y + z = k, \quad (3.8)$$

where  $x, y, z \in \{0, 1, \dots\}$  has only three solutions:

$$k = 7 : \quad x = 3, y = 0, z = 4; \quad x = 1, y = 6, z = 0; \quad x = 2, y = 3, z = 2.$$

$$k = 8 : \quad x = 2, y = 2, z = 4; \quad x = 0, y = 8, z = 0; \quad x = 1, y = 5, z = 2.$$

$$k = 9 : \quad x = 0, y = 7, z = 2; \quad x = 1, y = 4, z = 4; \quad x = 2, y = 1, z = 6.$$

Recall that, by (2.3),  $g(v, w^1) = 2^k$  if and only if the word  $v$  has  $k \geq 0$  letters  $b$ . We use (2.4), where the values  $g(v, w^1)$  are such as in the solutions of (3.8).

Let  $|V_{w^1}| = 5$  and  $|V_{w^2}| \geq 5$ . Then, by Corollary 3.4, there is  $i \in [4]$  such that  $v_i = b$  for every  $v \in V_{w^1}$ . If  $i \in \{1, 2, 3\}$ , then  $V_{w^1} \cap V_{w^2} = \emptyset$ , and consequently  $|V_{w^1} \cup V_{w^2}| \geq 10$ . Since  $V^{4, b'} \cap (V_{w^1} \cup V_{w^2}) = \emptyset$  and, by (A),  $|V^{4, b'}| \geq 2$ , we have  $|V| \geq 12$ .

If  $i = 4$ , then  $V_{w^1} \subset V^{4, b}$ . If  $|V_{w^2}| = 5$ , then we can assume, by just considered case that  $v_4 = b$  for all  $v \in V_{w^2}$ . Then  $|V^{4, b}| \geq 8$ . By (A),  $|V| > 12$ .

If  $|V_{w^2}| \geq 7$ , then observe that  $g(v, w^2) \geq 2$  for every  $v \in V_{w^2}$ , as if  $g(v, w^2) = 1$  for some  $v \in V_{w^2}$ , then, by (2.1),  $v \in V_{w^1}$  which is impossible. In view of the solutions of (3.8) we have: if  $|V_{w^2}| = 7$ , then  $|V_{w^1} \cap V_{w^2}| \leq 2$ , and if  $|V_{w^2}| = 8$ , then  $|V_{w^1} \cap V_{w^2}| \leq 2$ . In both cases we get  $|V_{w^1} \cup V_{w^2}| \geq 10$ , and then  $|V| \geq 12$ .

Let  $|V_{w^1}| = 7$  and  $|V_{w^2}| \geq 7$  and let  $V_{w^1} = \{v^1, \dots, v^7\}$ . Assume first that  $g(v^i, w^1) = 4$  for  $i \in \{1, 2, 3\}$  and  $g(v^i, w^1) = 1$  for  $i \in \{4, 5, 6, 7\}$  (the first solution for  $k = 7$ ). Then  $v^1, v^2, v^3 \notin V_{w^2}$ , and thus  $|V_{w^1} \cup V_{w^2}| \geq 10$ . Hence,  $|V| \geq 12$ , as  $|V^{4, b'}| \geq 2$ .

Let now  $g(v^1, w^1) = 4$  and  $g(v^i, w^1) = 2$  for  $i \in \{2, \dots, 7\}$ . Then  $v^1 \notin V_{w^2}$ . As we have just seen it is enough to show that  $|V_{w^1} \cup V_{w^2}| \geq 10$  or  $|V^{4, b}| \geq 6$ . Note that,  $v^i \in V_{w^2}$ , where  $i \in \{2, \dots, 7\}$ , if and only if  $v_4^i = b$ . Thus,  $v^i \in V_{w^2}$  for at most three  $i \in \{2, \dots, 7\}$  or for every  $i \in \{2, \dots, 7\}$ . In the first case, if on the contrary  $v^i \in V_{w^2}$  for at least four but less than six  $i \in \{2, \dots, 7\}$ , by Lemma 3.7, there is a twin pair in  $V_{w^1}$ , which is a contradiction. Then  $|V_{w^1} \cup V_{w^2}| > 10$ , and so  $|V| \geq 12$ . In the second case we have  $|V^{4, b}| \geq 6$ , and by (A),  $|V| \geq 12$ .

Let  $g(v^i, w^1) = 4$  for  $i \in \{1, 2\}$ ,  $g(v^i, w^1) = 2$  for  $i \in \{3, 4, 5\}$  and  $g(v^i, w^1) = 1$  for  $i \in \{6, 7\}$ . Then  $v^1, v^2 \notin V_{w^2}$ . Assume that  $v^3, v^4, v^5 \in V_{w^2}$ , i.e.  $v_4^3 = v_4^4 = v_4^5 = b$ . Clearly,  $v^6, v^7 \in V_{w^2}$ . The set  $\bigcup \{\check{w}^1 \cap \check{v} \neq \emptyset : v \in V_{w^1} \text{ } v_4 \in \{a, a'\}\}$  is a 4-cylinder in the 4-box  $\check{w}^1$ . If it contains at most three boxes, then, by Lemma 3.7, there is a twin pair in  $V_{w^1}$ , which is a contradiction. Thus, there are four boxes in the above cylinder. These are:  $\check{w}^1 \cap \check{v}^i$  for  $i \in \{1, 2, 6, 7\}$ . Since  $g(v^i, w^1) = 4$  for  $i \in \{1, 2\}$  and  $g(v^i, w^1) = 1$  for  $i \in \{6, 7\}$ , it must be  $v_4^1 = v_4^2 = a$  and  $v_4^6 = v_4^7 = a'$ . By Lemma 3.2, the set  $\check{w}^1 \cap \check{v}^1 \cup \check{w}^1 \cap \check{v}^6$  is a  $L$ -polybox in  $\check{w}^1$ . But this is impossible as  $v^1$  and  $v^6$  are not of the form predicted in Lemma 3.2, where  $s = w^1$ , i.e. the word  $v^1$  contains two letters  $b$ ,

and  $v^6$  does not contain a letter  $b$ . Therefore  $v^i \notin V_{w^1}$  for some  $i \in \{3, 4, 5\}$ , and thus  $|V_{w^1} \cup V_{w^2}| \geq 10$  which gives  $|V| \geq 12$ .

Let  $|V_{w^1}| = 8$  and  $|V_{w^2}| \geq 8$  and let  $V_{w^1} = \{v^1, \dots, v^8\}$ . Clearly, it is enough to show that  $|V_{w^1} \cup V_{w^2}| \geq 10$  or  $|V^{4,b}| \geq 6$ .

If  $g(v^i, w^1) = 4$  for  $i \in \{1, 2\}$ , then  $v^1, v^2 \notin V_{w^2}$ . Thus,  $|V_{w^1} \cup V_{w^2}| \geq 10$ .

If  $g(v^i, w^1) = 2$  for  $i \in \{1, \dots, 8\}$  and  $v^i \notin V_{w^2}$  for at most one  $i \in \{1, \dots, 8\}$ , then  $|V^{4,b}| \geq 7$ . Otherwise,  $|V_{w^1} \cup V_{w^2}| \geq 10$ .

Let  $g(v^1, w^1) = 4$ ,  $g(v^i, w^1) = 2$  for  $i \in \{2, 3, 4, 5, 6\}$  and  $g(v^i, w^1) = 1$  for  $i \in \{7, 8\}$ . Then  $v^1 \notin V_{w^2}$ . For the same reason as in the last case for  $|V_{w^1}| = 7$ , by Lemma 3.7,  $v^i \in V_{w^2}$  for at most four  $i \in \{2, \dots, 6\}$ , and therefore  $v^i \notin V_{w^2}$  for some  $i \in \{2, \dots, 6\}$ . Consequently,  $|V_{w^1} \cup V_{w^2}| \geq 10$ .

Let  $|V_{w^1}| = 9$  and  $|V_{w^2}| = 9$ . Now we have to show that there is at least one words in  $V_{w^1}$  which is not an element of  $V_{w^2}$  or  $|V^{4,b}| \geq 6$ . But this obvious in a view of the solutions of (3.8) for  $k = 9$ .

Let  $d \geq 5$ . By (A) we can assume that

$$|V^{i,l} \cup V^{i,s}| \leq 7, \quad (3.9)$$

for every  $l, s \in \{a, a', b, b'\}$ . Assume that there are  $i \in [d]$  and two letters, say  $a$  and  $b$ , such that there are no  $i$ -siblings  $u$  and  $v$  in  $V$  such that  $u_i = a$  and  $v_i = b$ . Then, by Lemma 3.10, the polybox

$$(\pi_x^i \cap \bigcup E(V))_i = (\pi_x^i \cap \bigcup E(V^{i,a} \cup V^{i,b}))_i,$$

where  $x \in Ea \cap Eb$ , is rigid. Therefore,  $(v)_i = (w)_i$  for some  $v \in V^{i,a}$  and  $w \in W^{i,b}$ . By Statement 3.8 (a),  $|V| \geq 12$ .

Thus, in what follows we assume that for every  $i \in [d]$  and every two letters  $l, s \in \{a, a', b, b'\}$  such that  $l \notin \{s, s'\}$  there are  $i$ -siblings  $u$  and  $v$  in  $V$  such that  $u_i = l$  and  $v_i = s$ . This means that there are at least  $4d$  edges in the set  $\mathcal{E}$ .

Let  $u^0, v^0 \in V$  be such that

$$d(v^0) + d(u^0) = \max\{d(v) + d(u) : v, u \in V \text{ and } v, u \text{ are adjacent}\}.$$

Let  $d = 5$ . If  $d(v^0) + d(u^0) \geq 9$  then, from (3.6) or (3.7) it follows that there are  $i \in [d]$  and  $l \in S$  such that  $|V^{i,l} \cup V^{i,l'}| \geq 8$ , which contradicts (3.9).

Let  $d(v^0) + d(u^0) = 8$ , and let  $N(u^0), N(v^0)$  be the sets of all neighbors of  $u^0$  and  $v^0$ , respectively. Taking into account (3.9), it can be checked that there are  $j, k \in [5], k \neq j$ , and  $l, s \in \{a, b\}$  such that

$$|(V^{j,l} \cup V^{j,l'}) \cap (N(v^0) \cup N(u^0))| = 7$$

and

$$|(V^{k,s} \cup V^{k,s'}) \cap (N(v^0) \cup N(u^0))| = 7.$$

Without loss of generality we can take  $l = s = a$ , because  $k \neq j$ .

Since  $|V^{j,b} \cup V^{j,b'}| \geq 4$  and  $|V^{k,b} \cup V^{k,b'}| \geq 4$ , there are at least three words  $x, y, z$  in the set  $(V^{j,b} \cup V^{j,b'}) \cap (V \setminus (N(u^0) \cup N(v^0)))$  and at least three words  $\bar{x}, \bar{y}, \bar{z}$  in the set  $(V^{k,b} \cup V^{k,b'}) \cap (V \setminus (N(u^0) \cup N(v^0)))$ . If  $\{x, y, z\} \neq \{\bar{x}, \bar{y}, \bar{z}\}$ , then  $|V| \geq 12$ . Let us assume that  $\{x, y, z\} = \{\bar{x}, \bar{y}, \bar{z}\}$  and  $|V| = 11$ . Then  $x_j, y_j, z_j \in \{b, b'\}$  and  $x_k, y_k, z_k \in \{b, b'\}$ . On the other hand, the vertices  $u \in (N(v^0) \cup N(u^0)) \setminus (V^{j,a} \cup V^{j,a'})$  and  $v \in (N(v^0) \cup N(u^0)) \setminus (V^{k,a} \cup V^{k,a'})$  are such

that  $u_j \in \{b, b'\}$  and  $v_k \in \{b, b'\}$ . Assume that  $u \neq v$ . Since  $w_j, w_k \in \{a, a'\}$  for every  $w \in (N(v^0) \cup N(u^0)) \setminus \{u, v\}$ , it follows that if  $x, y$  and  $z$  are joined with some vertices from the set  $N(u^0) \cup N(v^0)$ , then these vertices must be  $u$  or  $v$ . This, together with the facts that  $d(v^0) + d(u^0) = 8$ ,  $d(v) \leq 5$  for every  $v \in V$  and the graph  $G(V, E)$  does not contain triangles, imply that the number of edges with ends in the set  $V = N(u^0) \cup N(v^0) \cup \{x, y, z\}$  is less than 20: if  $|N(u^0)| = 5$ ,  $|N(v^0)| = 3$  and  $u \in N(u^0)$ ,  $v \in N(v^0)$ , then  $|\mathcal{E}| \leq 18$ , and if  $u, v \in N(u^0)$  or  $u, v \in N(v^0)$ , then  $|\mathcal{E}| \leq 17$ ; if  $|N(u^0)| = |N(v^0)| = 4$  and  $u \in N(u^0)$ ,  $v \in N(v^0)$ , then  $|\mathcal{E}| \leq 19$ , and if  $u, v \in N(u^0)$ , then  $|\mathcal{E}| \leq 18$ . Similarly, if  $u = v$ , then there are less than 20 edges in  $\mathcal{E}$ . Since there are at least 20 edges in the graph  $G$ , it follows that  $|V| \geq 12$ .

If  $d(v^0) + d(u^0) \leq 7$ , then from Lemma 3.9 we have  $d(G) \leq 7/2$ . As  $d(G)|V| = 2|\mathcal{E}|$  and  $2|\mathcal{E}| \geq 40$ , we have  $|V| > 11$ .

Let  $d \geq 6$ . If  $d(v^0) + d(u^0) \geq 9$ , then using (3.6) or (3.7), in the same way as for  $d = 5$  we show that  $|V| \geq 12$ . If  $d(v^0) + d(u^0) \leq 8$ , then from Lemma 3.9 it follows that  $d(G) \leq 4$ . Since now  $2|\mathcal{E}| \geq 48$ , we have  $|V| \geq 12$ .  $\square$

**REMARK 3.1** The estimation given in Theorem 3.1 is optimal. There are two equivalent polybox codes  $V, W \subset \{a, a', b, b'\}^4$  both without twin pairs and  $V \cap W = \emptyset$  ([14]). These codes were used by Lagarias and Shor[13] and later on by Mackey[17] to construct the counterexamples to Keller's cube tiling conjecture. In the context of this conjecture one of these codes was given first by Corrádi and Szabó in [3], as an example of the maximal clique in a 4-dimensional Keller graph.

## 4 Twin pairs in cube tilings of $\mathbb{R}^d$

From Theorem 3.1 we obtain the following result.

**COROLLARY 4.1** *Let  $W = W^{i, l_1} \cup W^{i, l'_1} \cup \dots \cup W^{i, l_k} \cup W^{i, l'_k} \subset S^d$  be a partition code, where  $W^{i, l} = \{w \in W : w_i = l\}$  for  $l \in S$  and  $i \in [d]$ . If  $k > 2^{d-3}/3$ , then there is a twin pair in  $W$ .*

*Proof.* Let us recall that for every  $j \in [k]$  the polybox codes  $(W^{i, l_j})_i \subset S^{d-1}$  and  $(W^{i, l'_j})_i \subset S^{d-1}$  are equivalent. By the assumption on the number  $k$ , there is at least one  $j \in [k]$  such that

$$|(W^{i, l_j})_i| \leq 11.$$

If the polybox codes  $(W^{i, l_j})_i$  and  $(W^{i, l'_j})_i$  do not contain twin pairs, then, by Theorem 3.1, they are equal, and therefore the code  $W^{i, l_j} \cup W^{i, l'_j}$  consists of twin pairs. If  $(W^{i, l_j})_i$  contains a twin pair  $(w)_i, (v)_i$ , then  $W^{i, l_j}$  contains the twin pair  $w = w_1 \dots w_{i-1} l_j w_{i+1} \dots w_d$ ,  $v = v_1 \dots v_{i-1} l_j v_{i+1} \dots v_d$ .  $\square$

Thus we have the following theorem on twin pairs in cube tilings of  $\mathbb{R}^d$ .

**THEOREM 4.2** *Let  $[0, 1]^d + T$  be a cube tiling of  $\mathbb{R}^d$ ,  $x \in \mathbb{R}^d$  and  $i \in [d]$ . Let  $L(T, x, i)$  be the set of all  $i$ th coordinates  $t_i$  of vectors  $t \in T$  such that  $([0, 1]^d + t) \cap ([0, 1]^d + x) \neq \emptyset$  and  $t_i \leq x_i$ . If there are  $x \in \mathbb{R}^d$  and  $i \in [d]$  such that  $|L(T, x, i)| > 2^{d-3}/3$ , then there is a twin pair in the tiling  $[0, 1]^d + T$ .*

*Proof.* In Section 1 we showed that the family of boxes  $\mathcal{F}_x = \{([0, 1]^d + x) \cap ([0, 1]^d + t) \neq \emptyset : t \in T\}$  is a minimal partition of the  $d$ -box  $[0, 1]^d + x$ . Let  $W = W^{i, l_1} \cup W^{i, l'_1} \cup \dots \cup W^{i, l_k} \cup W^{i, l'_k}$  be a partition code such that  $\mathcal{F}_x$  is an exact realization of  $W$ . Note that  $|L(T, x, i)| = k$ , i.e.  $|L(T, x, i)|$  is the number of all  $i$ -cylinders in the partition  $\mathcal{F}_x$ . Indeed,  $t_i \in L(T, x, i)$  if and only if there is  $t' \in T$  such that  $([0, 1]^d + t') \cap ([0, 1]^d + x) \neq \emptyset$  and  $t'_i - t_i = 1$ . By Corollary 4.1, there is a twin pair in  $\mathcal{F}_x$ , and thus there is a twin pair in the tiling  $[0, 1]^d + T$ .  $\square$

**COROLLARY 4.3** *Keller's conjecture is true for a cube tiling  $[0, 1]^7 + T$  of  $\mathbb{R}^7$  for which there are  $x \in \mathbb{R}^d$  and  $i \in [d]$  such that the set  $L(T, x, i)$  contains at least six elements.*  $\square$

**COROLLARY 4.4** *If there is a counterexample to Keller's conjecture in dimension seven, then  $|L(T, x, i)| \in \{3, 4, 5\}$  for some  $x \in \mathbb{R}^7$  and  $i \in [7]$ .*

*Proof.* Let  $[0, 1]^7 + T$  be a cube tiling of  $\mathbb{R}^7$ . By Corollary 4.3, we assume that  $|L(T, x, i)| \leq 2$  for some  $x \in \mathbb{R}^7$  and every  $i \in [7]$ . (We will use somewhat different arguments to show that there is a twin pair in  $[0, 1]^7 + T$ , than that used in Section 1.) Thus, the minimal partition  $\mathcal{F}_x = \{([0, 1]^d + x) \cap ([0, 1]^d + t) \neq \emptyset : t \in T\}$  contains at most two  $i$ -cylinders for every  $i \in [7]$ , and therefore its partition code  $V$  can be written in the alphabet  $S = \{a, a', b, b'\}$ . If we take  $a = 0$ ,  $a' = 2$ ,  $b = 1$  and  $b' = 3$ , then, as it was shown in [4], the maximal clique in a Keller graph has 124 vertices. Thus, there is a twin pair in  $V$ , and consequently there is a twin pair in  $[0, 1]^7 + T$ .  $\square$

We extend the notion of a  $d$ -dimensional Keller graph. If  $S$  is an alphabet with a complementation, then a  $d$ -dimensional Keller graph on the set  $S^d$  is the graph in which two vertices  $u, v \in S^d$  are adjacent if they are dichotomous but do not form a twin pair. This extension lies in the fact that, now the set of vertices is  $S^d$ , where  $S$  an arbitrary alphabet with a complementation, while in a  $d$ -dimensional Keller graph we have  $|S| = 4$  ([3]). Indeed, the condition that in a  $d$ -dimensional Keller graph for every two adjacent vertices  $u, v$  there is  $i \in [d]$  such that  $u_i$  and  $v_i$  differs by two modulo four means that  $v$  and  $u$  are dichotomous, where complementation is given by  $0 = 2'$  and  $1 = 3'$ ; the condition that  $u_j \neq v_j$  for some  $j \neq i$  means that  $v$  and  $u$  are not a twin pair. From Corollary 4.1 we obtain the following

**COROLLARY 4.5** *Every clique in a  $d$ -dimensional Keller graph on  $S^d$  which contains at least  $k > 2^{d-3}/3$  vertices  $u^1, \dots, u^k$  such that  $u_i^n \notin \{u_i^m, (u_i^m)'\}$  for some  $i \in [d]$  and every  $n, m \in \{1, \dots, k\}, n \neq m$ , has less than  $2^d$  elements. In particular, any clique in a 7-dimensional Keller graph on  $S^7$  which contains at least six vertices  $u^1, \dots, u^6$  such that  $u_i^n \notin \{u_i^m, (u_i^m)'\}$  for some  $i \in [7]$  and every  $n, m \in \{1, \dots, 6\}, n \neq m$ , has less than 128 elements.*  $\square$

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